On two tests for exponentiality with different motivations, and power results for EDF tests

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Classical tests: Empirical Distribution Function (EDF)

- EDF statistics: definitions

- the EDF of a sample $x_1, x_2, \ldots, x_n$ is

$$F_n(t) = \frac{\{\#x_i \leq t\}}{n} \text{ for } -\infty \leq t \leq \infty.$$

- statistics based on the discrepancy

$$Z_n(x) = F_n(x) - F(x;\theta)$$

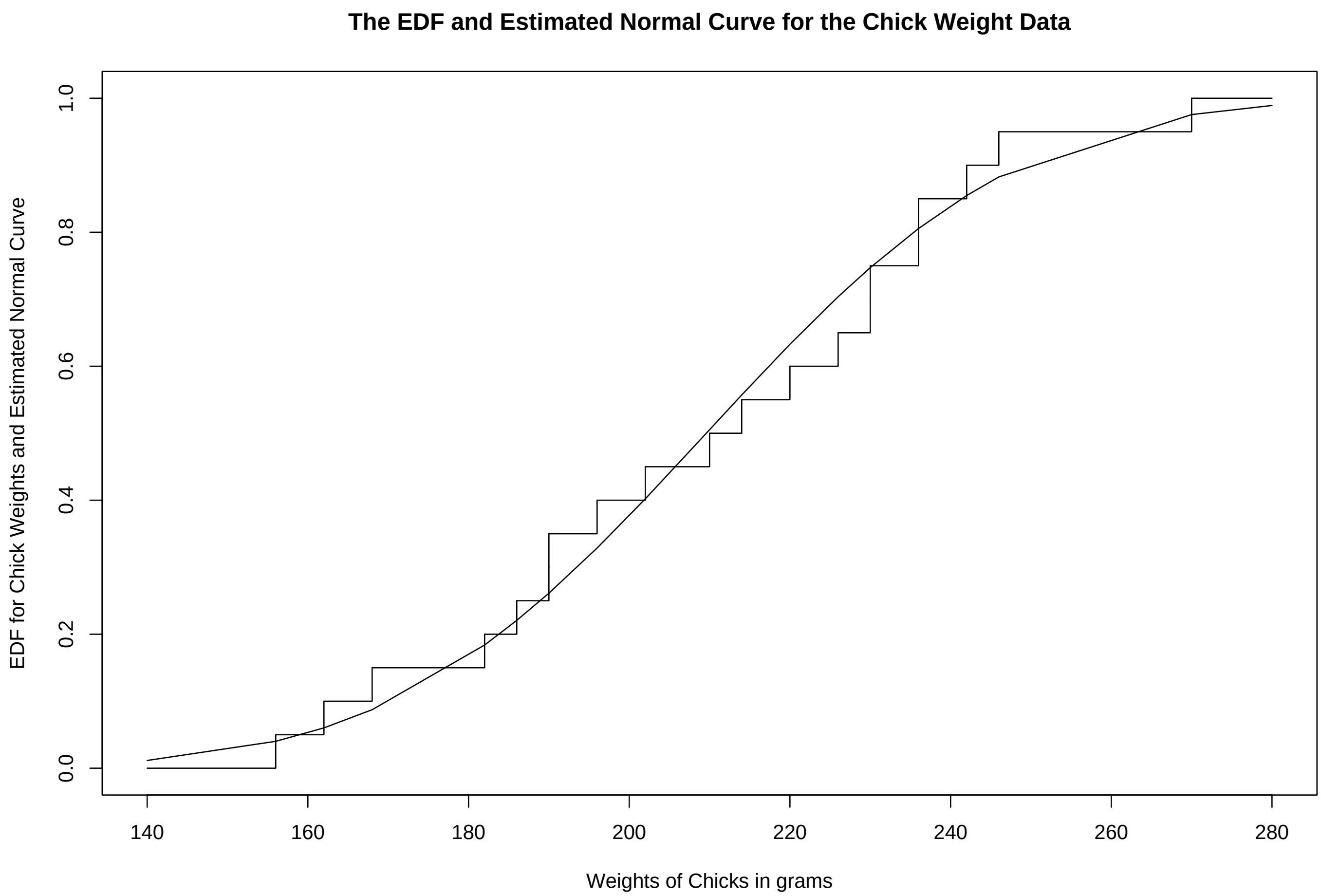
- most well known are

$D = \max|Z(x)|$

$W^2 = n\int_{-\infty}^{\infty} Z^2(x) dF(x;\theta)$

$A^2 = n\int_{-\infty}^{\infty} Z^2(x)\psi(x) dF(x;\theta)$

where $\psi(x) = [F(x;\theta)(1 - F(x;\theta))]^{-1}$


The EDF and Estimated Normal Curve for the Chick Weight Data
EDF for Chick Weights and Estimated Normal Curve
0.0
0.2
0.4
0.6
0.8
1.0
140
160
180
200
220
240
260
280
Weights of Chicks in grams

EDF statistics : the PIT

- Probability Integral Transformation (PIT)

$$u_{(i)} = F(x_{(i)}; \theta), \quad i = 1, \ldots, n$$

When θ known, u-set is uniform $U(0,1)$; so compare $F_n(u)$ with $F(u) = u$, $0 < u < 1$.

- For the Exp distribution, θ unknown, PIT is:

$$u_i = 1 - \exp(-x_i/\bar{x})$$

- Cramér von Mises statistic W^2 and Anderson–Darling statistic A^2 are

$$\begin{aligned} W^2 &= \sum_{i=1}^{n} \left\{ u_{(i)} - \frac{2i-1}{2n} \right\}^2 + \frac{1}{12n} \text{ and} \\ A^2 &= -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \ln\left(u_{(i)}\right) + \right. \\ &\qquad \left. + \ln\left(1 - u_{(n+1-i)}\right) \right\} \end{aligned}$$

Total time on test statistics

- Light bulbs put on test at time $x_0 = 0$; $x_{(1)}, x_{(2)}, \cdots$ ordered times to failures.

- Define $S_i = (n+1-i)x_{(i)} + \sum_{k=1}^{i-1} x_{(k)}$.

- Let $v_{(i)} = S_i/T, \; i = 1, \ldots, n$. The $v_{(i)}$, $i = 1, \ldots, n-1$, are order statistics from $U(0,1)$ ($v_{(n)} = 1$), so EDF tests used to test uniformity.

- Table 10.6 in Stephens (1986a) shows a remarkable similar powers, classical and TTT The $u_{(i)}$ and $v_{(i)}$ must be 'close'.

Closeness of u and v

- Define *closeness*:
  Let $C_n(x,y) \equiv \sum_{i=1}^{n} \left(u_{(i)} - v_{(i)}\right)^2$.
  Closeness is:

  $$C(x,y) = \lim_{n\to\infty} E\left\{C_n(x,y)\right\}.$$

- Closeness of ordered uniforms to their means $m_i = i/(n+1)$: $C(v,m) = 1/6$.

- Explore large-sample behaviour of $u_{(i)}$ and $v_{(i)}$; derived as above, for testing the exponential distribution, and keep terms only to $1/\sqrt{(}n)$.

- Using $W(t)$, (Brownian Bridge) covariance

  $$\rho_0(s,t) = \min(s,t) - st.$$

Closeness on contiguous alternatives

- Contiguous alternatives - approaching the exponential as $1/\sqrt{n}$.

- 1. Weibull W(β), density

$$f(x) = \beta \left\{ x^{\beta-1}/\theta^{\beta} \right\} \exp(-x/\theta)^{\beta},\ x \geq 0.$$

- 2. Gamma(β), density

$$f(x) = \frac{x^{\beta-1}}{\theta^{\beta}\Gamma(\beta)} \exp(-x/\theta),\ x \geq 0:$$

  let $\beta = 1 + \gamma/\sqrt{n}$.

Use of the Quantile Process

- Suppose $Q_n(t)$ is the quantile process for the exponential distribution.

- Let $j = i/(n+1)$; then
$$Q_n(j) = \sqrt{n}\left\{x_{(i)} - F^{-1}(j)\right\}.$$

- Asymptotically, $Q_n(t) \rightarrow Q(t)$, a Gaussian process with mean zero and covariance
$$\rho(s,t) = \frac{\min(s,t) - st}{(1-s)(1-t)}.$$

- We have
$$u_{(i)} = 1 - \exp(-x_{(i)}/\bar{x})$$
and must get an expression for $u_{(i)}$, keeping terms only to $1/\sqrt{(n)}$. Similarly for $v_{(i)}$.

Closeness of u and v: Weibull alt

- Let $A(j) = (1-j)\ln(1-j)$,

$$B(j) = \ln(1-j)\ln\{-\ln(1-j)\}$$

and $C(j) = \int_0^{-\ln(1-j)} e^{-t}\ln(t)\,dt$.

- For the classical EDF tests

$$\begin{aligned} u_{(i)} &= j + \frac{1}{\sqrt{n}}W(j) + A(j)\int_0^1 \frac{W(v)}{1-v}dv \\ &\quad + \frac{\gamma}{\sqrt{n}}\left\{(1-j)B(j) - \Gamma^{'}(2)A(j)\right\}. \end{aligned}$$

- For the total time on test statistics

$$\begin{aligned} v_{(i)} &= j + \frac{1}{\sqrt{n}}\{W(j) + \int_0^j \frac{W(s)}{1-s}\,ds \\ &\quad -j\int_0^1 \frac{W(s)}{1-s}\,ds\} + \frac{\gamma}{\sqrt{n}}\{(1-j)B(j) \\ &\quad -C(j) + j\Gamma^{'}(2)\} \end{aligned}$$

Closeness of u and v

- The first term of the stochastic components of $u(i) - j$ and $v(i) - j$ are the same: also the first terms of the non-random components.

- Then after calculations:

$$C_W(u, m; \gamma) = 0.0559\gamma^2 + 5/54$$

$$C_W(v, m; \gamma) = 0.1391\gamma^2 + 9/54$$

$$C_W(u, v; \gamma) = 0.0416\gamma^2 + 2/54$$

- Similar calculations give closeness for the Gamma alternative:

$$C_G(u, m; \gamma) = .0190\gamma^2 + 5/54$$

$$C_G(v, m; \gamma) = .3903\gamma^2 + 9/54$$

$$C_G(u, v; \gamma) = .2565\gamma^2 + 2/54$$

Un peu d'histoire

- Anderson and Darling (1952), Darling(1955), Sukhatme (1974), MAS (1976) investigated asymptotic distributions of EDF statistics by going straight to the asymptotic Gaussian process $Z(x)$ of

$$Z_n(x) = F_n(x) - F(x;\theta)$$

  for both θ known or estimated efficiently.

- Durbin and Knott (1972), Durbin, Knott and Taylor (1975), MAS (1974) expanded $Z_n(x)$ as a Fourier series (and the corresponding process for A^2 using Legendre functions) and so looked at the finite-n case.

- Finally, RAL and MAS have looked at the behaviour of the individual order statistics.

Un peu d'histoire, cont.

- On the null, asymptotic distributions of the form

$$W_\infty^2 = \sum_{j=1}^{\infty} Z_j^2/\lambda_j$$

  where Z_j are independent standard normal variables. The finite-n versions are called components by DK, DKT.

- For contiguous alternatives, the asymptotic distributions are as above (same λ_j as on the null), but now Z_j are independent normals, variance 1, but with mean $\gamma\delta_j$.

- DK, DKT pushed the components as test statistics; recall Neyman tests. They showed how to get δ_j, but made errors.

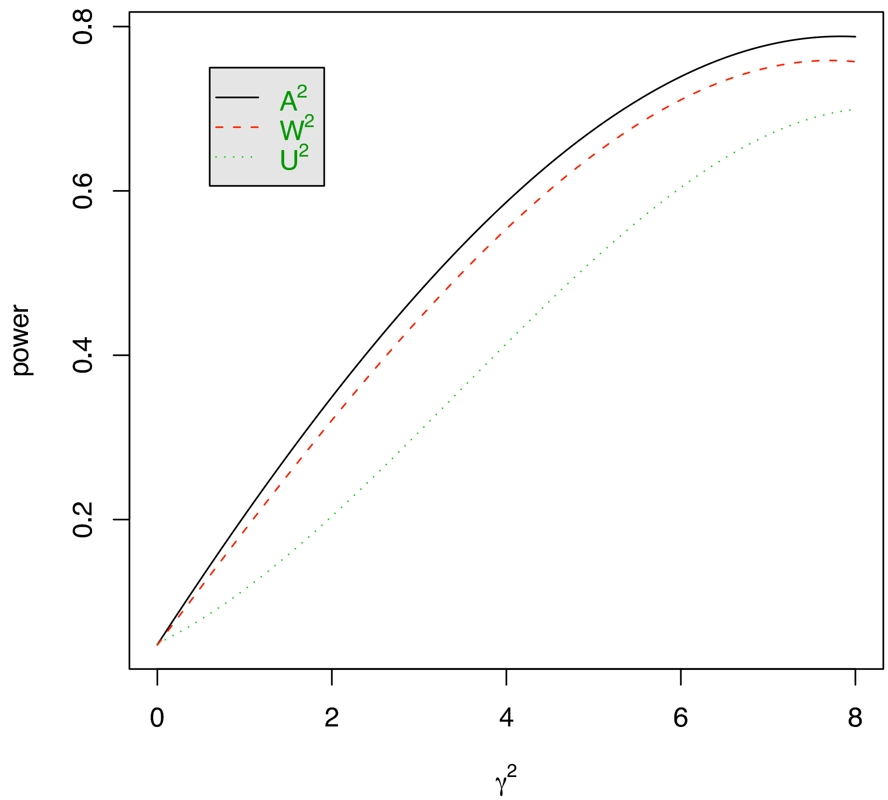
Asymptotic power of EDF statistics

- In power studies for tests of fit, it has often been shown thet the Anderson–Darling statistic has high power.

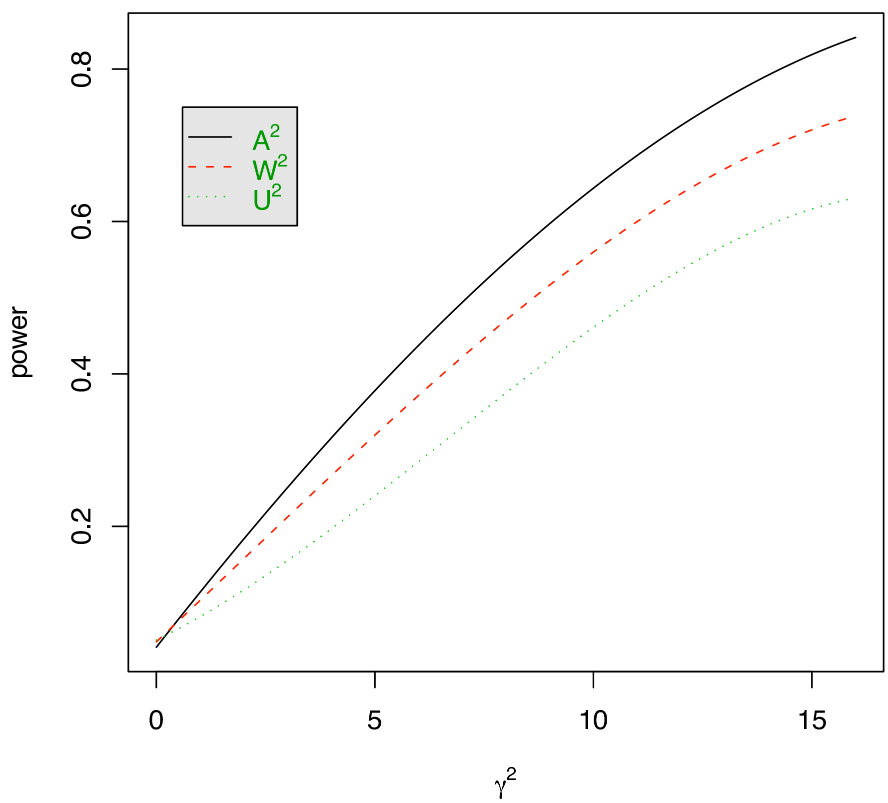
- In Stephens (1981), I followed the Durbin–Knott technique to discuss asymptotic power against contiguous Weibull or Gamma alternatives. Asymptotic powers will be functions of γ and can be compared.

- The following plots show the power against γ^2, for 5% tests, when the alternative is Weibull or Gamma.

Power against the Weibull alternative



Power against the Gamma alternative



It may be seen that, appropriately for this celebration, the Anderson–Darling statistic once again tops the power plots.