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NOTE

A Note on a Vector-Variate Normal Distribution and a Stationary Autoregressive Process

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It is shown that weak stationarity of a first-order autoregressive process implies that eigenvalues of the coefficient matrix are less than 1 in absolute value.

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Nguyen (1997) has shown (Theorem 2.1) that if $\mathbf{X}_1$ and $\mathbf{X}_2$ are identically distributed random vectors such that

$$\mathbf{X}_2 = \mathbf{B}\mathbf{X}_1 + \mathbf{U}_2, \tag{1}$$

$\mathbf{U}_2$ and $\mathbf{X}_1$ are independent, and $\mathbf{U}_2$ has the distribution $N(\mathbf{0}, \mathbf{\Sigma})$ with $\mathbf{\Sigma}$ positive definite, then (a) the eigenvalues of $\mathbf{B}$ have modulus less than 1 and (b) $\mathbf{X}_1$ and $\mathbf{X}_2$ have a joint normal distribution with covariance matrix

$$\mathscr{E}\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}(\mathbf{X}_1', \mathbf{X}_2') = \begin{pmatrix}\mathbf{\Gamma} & \mathbf{B}\mathbf{\Gamma}\\ \mathbf{\Gamma}\mathbf{B}' & \mathbf{\Gamma}\end{pmatrix}, \tag{2}$$

where

$$\mathbf{\Gamma} = \sum_{s=0}^{\infty} \mathbf{B}^s \mathbf{\Sigma} \mathbf{B}'^s. \tag{3}$$

If the result is stated in the form of

$$\mathbf{X}_t = \mathbf{B}\mathbf{X}_{t-1} + \mathbf{U}_t, \tag{4}$$

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for $t=2$, it may be recognized as a form of the statement that a strictly stationary (autoregressive) process defined by (4) implies that the eigenvalues of $\mathbf{B}$ are less than 1 in absolute value and that if $\mathbf{U}_t$ is normal

$$\mathbf{X}_t = \sum_{s=0}^{\infty} \mathbf{B}^s \mathbf{U}_{t-s} \tag{5}$$

is Gaussian.

The purpose of this note is to show in a simple way that only stationarity in the wide sense needed for conclusion (a).

THEOREM. *Let $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{U}_2$ be related by* (1) *with $\mathbf{X}_1$ and $\mathbf{X}_2$ having the common covariance matrix $\boldsymbol{\Gamma}$, $\mathbf{U}_2$ having a nonsingular covariance matrix $\boldsymbol{\Sigma}$, and $\mathbf{X}_1$ and $\mathbf{U}_2$ uncorrelated. Then the eigenvalues of $\mathbf{B}$ are less than* 1 *in absolute value.*

Proof. An eigenvalue λ and eigenvector $\mathbf{x}$ satisfy

$$\mathbf{B}'\mathbf{x} = \lambda \mathbf{x}. \tag{6}$$

Then $\boldsymbol{\Gamma} = \mathbf{B}\boldsymbol{\Gamma}\mathbf{B}' + \boldsymbol{\Sigma}$ implies

$$\mathbf{x}'\boldsymbol{\Gamma}\bar{\mathbf{x}} = |\lambda|^2\, \mathbf{x}'\boldsymbol{\Gamma}\bar{\mathbf{x}} + \mathbf{x}'\boldsymbol{\Gamma}\bar{\mathbf{x}}. \tag{7}$$

Since $\mathbf{x}'\boldsymbol{\Sigma}\bar{\mathbf{x}} > 0$, (7) implies $\mathbf{x}'\boldsymbol{\Gamma}\bar{\mathbf{x}} > 0$ and $|\lambda|^2 < 1$. ∎

A sequence of random vectors $\mathbf{X}_t$ can be constructed recursively by (4), $t = 3, \ldots$. A consequence of the theorem is that (5) converges in the mean and $\{\mathbf{X}_t\}$ is stationary; if the $\mathbf{U}_t$ is independent of the $\mathbf{X}_{t-1}$, then $\{\mathbf{X}_t\}$ is Gaussian. See, for example, Anderson (1971, p. 179).

If $\mathbf{X}_t$ has mean $\mathscr{E}\mathbf{X}_t = \boldsymbol{\mu}$ possibly different from $\mathbf{0}$, then (1) is modified to $(\mathbf{X}_2 - \boldsymbol{\mu}) = \mathbf{B}(\mathbf{X}_1 - \boldsymbol{\mu}) + \mathbf{U}_2$ or (1) holds with $\mathbf{U}_2$ having the distribution $N(\boldsymbol{\nu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\nu} = (\mathbf{I} - \mathbf{B})\,\boldsymbol{\mu}$.

REFERENCES

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