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Reduced rank regression for blocks of simultaneous equations

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Abstract

Reduced rank regression analysis provides maximum likelihood estimators of a matrix of regression coefficients of specified rank and of corresponding linear restrictions on such matrices. These estimators depend on the eigenvectors of an "effect" matrix in the metric of an error covariance matrix. In this paper it is shown that the maximum likelihood estimator of the restrictions can be approximated by a function of the effect matrix alone. The procedures are applied to a block of simultaneous equations. The block may be over-identified in the entire model and the individual equations just-identified within the block. The procedures are generalizations of the limited information maximum likelihood and two-stage least squares estimators.

JEL classification: C13; C30

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1. Introduction

A simultaneous equation model (SEM) relates a set of endogenous or dependent variables to a set of exogenous or independent or predetermined variables with

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unobserved random or error variables. In contrast to many statistical studies the interest in SEM's is in linear restrictions on the regression of the dependent variables on the independent variables. In order to have nontrivial linear restrictions on the regression coefficients the regression matrix has to be of reduced rank. Reduced rank regression (RRR) and linear restrictions on regression matrices are two aspects of the one consideration.

A multivariate regression model is

$$\underset{T\times p}{\mathbf{Y}} = \underset{T\times q}{\mathbf{Z}}\ \underset{q\times p}{\mathbf{\Pi}} + \underset{T\times p}{\mathbf{V}}, \tag{1.1}$$

where the dependent matrix $\mathbf{Y}$ is $T \times p$, the independent matrix $\mathbf{Z}$ is $T \times q$, and the unobservable disturbance matrix $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_T)'$ is $T \times p$. Let $\mathscr{E}\mathbf{v}_t = \mathbf{0}$, $\mathscr{E}\mathbf{v}_t\mathbf{v}_t' = \mathbf{\Omega}$, and $\mathscr{E}\mathbf{v}_t\mathbf{v}_s' = \mathbf{0}$, $t \neq s$ [for example, Anderson (2003, Chapter 8)].

If rank $(\mathbf{\Pi}) = k < p, q$, the matrix $\mathbf{\Pi}$ can be written

$$\mathbf{\Pi} = \mathbf{GC}, \tag{1.2}$$

where $\mathbf{G}$ $(q \times k)$ and $\mathbf{C}$ $(k \times p)$ are suitable matrices of rank k. Alternatively the rank condition on $\mathbf{\Pi}$ can be expressed as

$$\mathbf{\Pi B} = \mathbf{0} \tag{1.3}$$

for suitable $\mathbf{B}$ $(p \times n)$ of rank n, where $n = p - k$.

Inference concerning $\mathbf{\Pi}$ of rank k and $\mathbf{\Omega}$ based on a sample $(\mathbf{Y}, \mathbf{Z})$ was developed by Anderson (1951). In particular the maximum likelihood estimators of $\mathbf{G}, \mathbf{C}, \mathbf{B}$ and $\mathbf{\Omega}$ were derived for the model of $\mathbf{v}_1, \ldots, \mathbf{v}_T$ being independently normally distributed. Later this analysis was called reduced rank regression (Izenman, 1975).

The case of one linear restriction was treated by Anderson and Rubin (1949, 1950). The present paper gives generalizations to several linear restrictions. Anderson and Rubin derived the asymptotic distribution of a maximum likelihood estimator of a linear restriction [the so-called limited information maximum likelihood (LIML) estimator] by finding the asymptotic distribution of the approximating simpler estimator [later termed the two-stage least squares (TSLS) estimator] [see Anderson (2005)]. These results are generalized here to the case of several restrictions.

2. Reduced rank regression

Denote a sample of T observations by $(\mathbf{y}_1, \mathbf{z}_1), \ldots, (\mathbf{y}_T, \mathbf{z}_T)$. Define $\mathbf{Y} = (\mathbf{y}_1, \ldots, \mathbf{y}_T)'$ and $\mathbf{Z} = (\mathbf{z}_1, \ldots, \mathbf{z}_T)'$. The second-order sample moments are

$$\mathbf{M}_{yy} = \frac{1}{T}\mathbf{Y}'\mathbf{Y}, \quad \mathbf{M}_{yz} = \frac{1}{T}\mathbf{Y}'\mathbf{Z}, \quad \mathbf{M}_{zz} = \frac{1}{T}\mathbf{Z}'\mathbf{Z}. \tag{2.1}$$

Because the interest is in the relations between $\mathbf{y}_t$ and $\mathbf{z}_t$, we treat these variables as if the means are $\mathbf{0}$. The maximum likelihood estimators of $\mathbf{\Pi}$ and $\mathbf{\Omega}$ when the rank of $\mathbf{\Pi}$ is unrestricted are

$$\mathbf{P} = \mathbf{M}_{zz}^{-1}\mathbf{M}_{zy} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y},$$
$$\mathbf{W} = \mathbf{M}_{yy} - \mathbf{P}'\mathbf{M}_{zz}\mathbf{P} = \frac{1}{T}(\mathbf{Y}'\mathbf{Y} - \mathbf{Y}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}). \tag{2.2}$$

The maximum likelihood estimators when the rank of $\mathbf{\Pi}$ is restricted involves the eigenvalues and eigenvectors of $\mathbf{P}'\mathbf{M}_{zz}\mathbf{P} = \frac{1}{T}\mathbf{Y}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}$ in the metric of $\mathbf{W}$. Let $\lambda_1 > \cdots > \lambda_p$ be the roots of

$$|\mathbf{P}'\mathbf{M}_{zz}\mathbf{P} - \lambda\mathbf{W}| = 0, \tag{2.3}$$

and let $\mathbf{b}_i$ be the solution to

$$(\mathbf{P}'\mathbf{M}_{zz}\mathbf{P} - \lambda_i\mathbf{W})\mathbf{b} = \mathbf{0}, \quad \mathbf{b}'\mathbf{W}\mathbf{b} = 1, \ \ i = 1, \ldots, p. \tag{2.4}$$

Define

$$\mathbf{F} = (\mathbf{b}_1, \ldots, \mathbf{b}_k), \quad \widetilde{\mathbf{B}} = (\mathbf{b}_{k+1}, \ldots, \mathbf{b}_p). \tag{2.5}$$

The maximum likelihood estimator of $\mathbf{\Pi}$ of rank k is

$$\widehat{\mathbf{\Pi}}_k = \widehat{\mathbf{G}}\widehat{\mathbf{C}}, \tag{2.6}$$

where $\widehat{\mathbf{G}} = \mathbf{PF}$ and $\widehat{\mathbf{C}} = \mathbf{F}'\mathbf{W}$. A maximum likelihood estimator of $\mathbf{B}$ is $\widetilde{\mathbf{B}}$. Note that $\widehat{\mathbf{\Pi}}_k\widetilde{\mathbf{B}} = \mathbf{0}$. An estimator of $\mathbf{B}$ equivalent to $\widetilde{\mathbf{B}}$ is $\widetilde{\mathbf{B}}\mathbf{O}_n$, where $\mathbf{O}_n$ is any orthogonal matrix of order n.

The matrix $\widetilde{\mathbf{B}}$ satisfies

$$\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\widetilde{\mathbf{B}} = \mathbf{W}\widetilde{\mathbf{B}}\mathbf{\Lambda}_n, \tag{2.7}$$

$$\widetilde{\mathbf{B}}'\mathbf{W}\widetilde{\mathbf{B}} = \mathbf{I}_n, \tag{2.8}$$

where $\mathbf{\Lambda}_n$ is the diagonal matrix with diagonal elements $\lambda_{k+1}, \ldots, \lambda_p$. The matrix $\mathbf{X} = \widetilde{\mathbf{B}}\mathbf{O}_n$ satisfies

$$\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\mathbf{X} = \mathbf{W}\mathbf{X}(\mathbf{O}_n'\mathbf{\Lambda}_n\mathbf{O}_n) \tag{2.9}$$

and (2.8) with $\widetilde{\mathbf{B}}$ replaced by $\mathbf{X}$; the matrix $\mathbf{O}_n'\mathbf{\Lambda}_n\mathbf{O}_n$ has eigenvalues $\lambda_{k+1}, \ldots, \lambda_p$. The matrix $\mathbf{X} = \widetilde{\mathbf{B}}\mathbf{N}_n$, where $\mathbf{N}_n$ is any nonsingular matrix of order n, satisfies

$$\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\mathbf{X} = \mathbf{W}\mathbf{X}(\mathbf{N}_n^{-1}\mathbf{\Lambda}_n\mathbf{N}_n), \tag{2.10}$$

but not

$$\mathbf{X}'\mathbf{W}\mathbf{X} = \mathbf{I}_n. \tag{2.11}$$

Note that $\mathbf{N}_n^{-1}\mathbf{\Lambda}_n\mathbf{N}_n$ has the eigenvalues $\lambda_{k+1}, \ldots, \lambda_p$.

Any matrix $\mathbf{B}$ of rank n satisfying $\mathbf{\Pi}\mathbf{B} = \mathbf{0}$ can be multiplied on the right by an arbitrary nonsingular matrix of order n to obtain another matrix satisfying $\mathbf{\Pi}\mathbf{B} = \mathbf{0}$. One way of eliminating this indeterminacy is requiring some $n \times n$ submatrix of $\mathbf{B}$ to be an assigned matrix. Suppose the rows of $\mathbf{B}$ are ordered so that $\mathbf{B} = (\mathbf{B}_1', \mathbf{B}_2')'$ with $\mathbf{B}_1$ square and nonsingular. Then $\mathbf{B}\mathbf{B}_1^{-1} = (\mathbf{I}_n, \mathbf{B}_1'^{-1}\mathbf{B}_2')'$. Thus we can require

$\mathbf{B}$ to have the form

$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_n \\ -\mathbf{A} \end{bmatrix}. \tag{2.12}$$

Correspondingly, let $\mathbf{\Pi} = (\mathbf{\Pi}_1, \mathbf{\Pi}_2)$. Then

$$\mathbf{0} = \mathbf{\Pi}\mathbf{B} = (\mathbf{\Pi}_1, \mathbf{\Pi}_2)\begin{pmatrix} \mathbf{I}_n \\ -\mathbf{A} \end{pmatrix} = \mathbf{\Pi}_1 - \mathbf{\Pi}_2\mathbf{A}; \tag{2.13}$$

that is, $\mathbf{\Pi}_1 = \mathbf{\Pi}_2\mathbf{A}$, and

$$\mathbf{\Pi} = \mathbf{\Pi}_2(\mathbf{A}, \mathbf{I}_k) = \mathbf{G}\mathbf{C}. \tag{2.14}$$

The (unique) maximum likelihood estimator of $\mathbf{A}$ is $\widehat{\mathbf{A}} = -\widetilde{\mathbf{B}}_2\widetilde{\mathbf{B}}_1^{-1}$, where $\widetilde{\mathbf{B}} = (\widetilde{\mathbf{B}}_1', \widetilde{\mathbf{B}}_2')'$.

3. An approximation to the maximum likelihood estimator

Now we shall find an approximation to the maximum likelihood estimator $\widehat{\mathbf{A}}$ that has the same asymptotic distribution. The outline of this development is similar to the use of the TSLS estimator as an approximation to the LIML estimator by Anderson and Rubin (1950). See also Anderson (2003, Section 12.7). Let $\operatorname{vec}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = (\mathbf{a}_1', \ldots, \mathbf{a}_m')'$, and let $(\mathbf{A} \otimes \mathbf{B}) = (a_{ij}\mathbf{B})$ be the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$.

Lemma 1. *Assume conditions on* $\{\mathbf{z}_t\}$ *and* $\{\mathbf{v}_t\}$ *such that*

$$\mathbf{M}_{zz} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{z}_t\mathbf{z}_t' \to \mathbf{M}_{zz}^0 \text{ nonsingular}, \tag{3.1}$$

$$\mathbf{P} \xrightarrow{p} \mathbf{\Pi}, \tag{3.2}$$

$$\sqrt{T}\operatorname{vec}(\mathbf{P} - \mathbf{\Pi}) = \operatorname{vec}\mathbf{P}^* \xrightarrow{d} N[\mathbf{0}, \mathbf{\Omega} \otimes (\mathbf{M}_{zz}^0)^{-1}], \tag{3.3}$$

$$\mathbf{W} \xrightarrow{p} \mathbf{\Omega}. \tag{3.4}$$

Then

$$\sqrt{T}\sum_{i=k+1}^{p} \lambda_i \xrightarrow{p} 0. \tag{3.5}$$

Proof. Since $\mathbf{b}_i'\mathbf{W}\mathbf{b}_i = 1$, $\mathbf{b}_i'\mathbf{W}\mathbf{b}_j = 0$, $i \neq j$, $\mathbf{b}_i'\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\mathbf{b}_i = \lambda_i$, and $\mathbf{b}_i'\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\mathbf{b}_j = 0$, $i \neq j$,

$$\sum_{i=k+1}^{p} \lambda_i = \sum_{i=k+1}^{p} \mathbf{b}_i'\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\mathbf{b}_i = \operatorname{tr}\widetilde{\mathbf{B}}'\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\widetilde{\mathbf{B}}, \tag{3.6}$$

$$\widetilde{\mathbf{B}}'\mathbf{W}\widetilde{\mathbf{B}} = \mathbf{I}_n. \tag{3.7}$$

For $\mathbf{C}$ $(p \times n)$

$$\sum_{i=k+1}^{p} \lambda_i = \min_{\substack{\mathbf{C} \\ \mathbf{C}'\mathbf{W}\mathbf{C}=\mathbf{I}_n}} \operatorname{tr} \mathbf{C}'\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\mathbf{C}. \tag{3.8}$$

This follows from the minimum–maximum properties of eigenvalues. For given $\boldsymbol{\alpha}_{i+1}, \ldots, \boldsymbol{\alpha}_p$

$$\min_{\substack{\boldsymbol{\alpha}_j'\mathbf{x}=0 \\ j=i+1,\ldots,p \\ \mathbf{x}'\mathbf{W}\mathbf{x}=1}} \mathbf{x}'\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\mathbf{x} \leqslant \lambda_i, \tag{3.9}$$

further

$$\min_{\substack{\mathbf{b}_j'\mathbf{W}\mathbf{x}=0 \\ j=i+1,\ldots,p \\ \mathbf{x}'\mathbf{W}\mathbf{x}=1}} \mathbf{x}'\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\mathbf{x} = \lambda_i. \tag{3.10}$$

Since $\boldsymbol{\Pi}$ has rank k, there is a matrix $\mathbf{B}\,(p \times n)$ of rank n such that $\boldsymbol{\Pi}\mathbf{B} = \mathbf{0}$ and $\mathbf{B}'\boldsymbol{\Omega}\mathbf{B} = \mathbf{I}_n$. Then

$$\begin{aligned}\sum_{i=k+1}^{p} \lambda_i &\leqslant \operatorname{tr} \mathbf{B}'\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\mathbf{B} \\ &= \operatorname{tr} \mathbf{B}'\left(\boldsymbol{\Pi}' + \frac{1}{\sqrt{T}}\mathbf{P}^{*'}\right)\mathbf{M}_{zz}\left(\boldsymbol{\Pi} + \frac{1}{\sqrt{T}}\mathbf{P}^{*}\right)\mathbf{B} \\ &= \frac{\operatorname{tr} \mathbf{B}'\mathbf{P}^{*'}\mathbf{M}_{zz}\mathbf{P}^{*}\mathbf{B}}{T}.\end{aligned} \tag{3.11}$$

Assumptions (3.1) and (3.3) imply that

$$\mathbf{B}'\mathbf{P}^{*'}\mathbf{M}_{zz}\mathbf{P}^{*}\mathbf{B} \xrightarrow{d} W(\mathbf{B}'\boldsymbol{\Omega}\mathbf{B}, q), \tag{3.12}$$

where $W(\mathbf{B}'\boldsymbol{\Omega}\mathbf{B}, q)$ denotes the Wishart distribution with covariance matrix $\mathbf{B}'\boldsymbol{\Omega}\mathbf{B}$ and q degrees of freedom. Then for $\mathbf{B}$ such that $\mathbf{B}'\boldsymbol{\Omega}\mathbf{B} = \mathbf{I}_n$, $\mathbf{B}'\mathbf{W}\mathbf{B} \xrightarrow{p} \mathbf{I}_p$,

$$\operatorname{tr} \mathbf{B}'\mathbf{P}^{*'}\mathbf{M}_{zz}\mathbf{P}^{*}\mathbf{B} \xrightarrow{d} \chi^2_{qn}, \tag{3.13}$$

and (3.5) follows. □

Since $\lambda_i \geqslant 0$, (3.5) implies that $\sqrt{T}\lambda_i \to 0$, $i = k+1, \ldots, p$. Consequences of Lemma 1 are that $\lambda_i\mathbf{W} = o_p(1/\sqrt{T})$, $\lambda_i\mathbf{W}\mathbf{b}_i = o_p(1/\sqrt{T})$, and $\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\mathbf{b}_i = o_p(1/\sqrt{T})$, $i = k+1, \ldots, p$. The last can be summarized as

$$\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}\widetilde{\mathbf{B}} = o_p(1/\sqrt{T}). \tag{3.14}$$

This fact suggests that $\widetilde{\mathbf{B}}$ can be approximated by a solution to

$$(\mathbf{P}'\mathbf{M}_{zz}\mathbf{P})_{n\cdot}\mathbf{B} = \mathbf{0}, \tag{3.15}$$

where $(\quad)_{n\cdot}$ denotes $(\quad)$ with the first n rows deleted. The $k \times p$ matrix $(\mathbf{P}'\mathbf{M}_{zz}\mathbf{P})_{n\cdot}$ could be replaced by a $k \times p$ matrix consisting of any k rows of $\mathbf{P}'\mathbf{M}_{zz}\mathbf{P}$.

Let $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2)$ where $\mathbf{P}_1$ has n columns and $\mathbf{P}_2$ has k columns. Then (3.15) is

$$\mathbf{0} = \mathbf{P}_2'\mathbf{M}_{zz}(\mathbf{P}_1, \mathbf{P}_2)\begin{bmatrix} \mathbf{I}_n \\ -\mathbf{A} \end{bmatrix} = \mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_1 - \mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_2\mathbf{A}. \tag{3.16}$$

The solution for $\mathbf{A}$ is

$$\widehat{\mathbf{A}}^{\mathrm{TS}} = (\mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_2)^{-1}\mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_1 = [\mathbf{Y}_2'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}_2]^{-1}\mathbf{Y}_2'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}_1, \tag{3.17}$$

where $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2)$. This is the TSLS estimator of $\mathbf{A}$. (The superscript TS denotes "two-stage.")

Theorem 1. *Under the conditions of Lemma* 1

$$\sqrt{T}\,\mathrm{vec}\,(\widehat{\mathbf{A}}^{\mathrm{TS}} - \mathbf{A}) \xrightarrow{d} N[\mathbf{0}, \mathbf{\Sigma}_{nn} \otimes (\mathbf{\Pi}_2\mathbf{M}_{zz}^0\mathbf{\Pi}_2')^{-1}], \tag{3.18}$$

where

$$\mathbf{\Sigma}_{nn} = \mathbf{B}'\mathbf{\Omega}\mathbf{B}. \tag{3.19}$$

Proof.

$$\begin{aligned}\widehat{\mathbf{A}}^{\mathrm{TS}} - \mathbf{A} &= (\mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_2)^{-1}(\mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_1 - \mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_2\mathbf{A}) \\ &= (\mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_2)^{-1}\mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}\mathbf{B} \\ &= (\mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_2)^{-1}\mathbf{P}_2'\mathbf{M}_{zy}\mathbf{B} \\ &= (\mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_2)^{-1}\mathbf{P}_2'\frac{1}{T}\sum_{t=1}^{T}\mathbf{z}_t\mathbf{u}_t',\end{aligned} \tag{3.20}$$

where $\mathbf{u}_t = \mathbf{B}'\mathbf{v}_t$ with $\mathscr{E}\mathbf{u}_t = \mathbf{0}$ and $\mathscr{E}\mathbf{u}_t\mathbf{u}_t' = \mathbf{B}'\mathbf{\Omega}\mathbf{B} = \mathbf{\Sigma}_{nn}$. □

Corollary 1. *Under the conditions of Lemma* 1

$$\sqrt{T}\,\mathrm{vec}\,(\widehat{\mathbf{A}} - \mathbf{A}) \xrightarrow{d} N[\mathbf{0}, \mathbf{\Sigma}_{nn} \otimes (\mathbf{\Pi}_2'\mathbf{M}_{zz}^0\mathbf{\Pi}_2)^{-1}]. \tag{3.21}$$

Proof. From (3.14) and $\widehat{\mathbf{B}} = (\mathbf{I}_n, -\widehat{\mathbf{A}}')'$ we find

$$\begin{aligned}\mathrm{o_p}\left(\frac{1}{\sqrt{T}}\right) &= \mathbf{P}_2'\mathbf{M}_{zz}(\mathbf{P}_1 - \mathbf{P}_2\widehat{\mathbf{A}}) \\ &= \mathbf{P}_2'\mathbf{M}_{zz}[\mathbf{P}_1 - \mathbf{P}_2\mathbf{A} - \mathbf{P}_2(\widehat{\mathbf{A}} - \mathbf{A})] \\ &= \mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_2(\widehat{\mathbf{A}}^{T} - \mathbf{A}) - \mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_2(\widehat{\mathbf{A}} - \mathbf{A})\end{aligned} \tag{3.22}$$

by the first line of (3.20). Then the corollary follows from Theorem 1. □

Since $\mathbf{P} \xrightarrow{p} \mathbf{\Pi}$ and $\mathbf{\Pi}$ has rank k, the components of $\mathbf{y}_t$ can be numbered so that $\mathbf{\Pi}_2$ has rank k. Hence, the probability that $\mathbf{P}_2'\mathbf{M}_{zz}\mathbf{P}_2$ is nonsingular approaches 1.

4. Inference in simultaneous equations models

A linear simultaneous equations model can be written in the form

$$\mathbf{YB} = \mathbf{Z\Gamma} + \mathbf{U}, \tag{4.1}$$

where $\mathbf{Y}$ is $T \times G$, $\mathbf{Z}$ is $T \times K$, $\mathbf{U} = T \times G$, $\mathbf{B}\,(G \times G)$ is nonsingular, and $\mathbf{\Gamma}$ is $K \times G$. The *reduced form* of this SEM is $\mathbf{Y} = \mathbf{Z\Pi} + \mathbf{V}$, where $\mathbf{\Pi} = \mathbf{\Gamma B}^{-1}$ and $\mathbf{V} = \mathbf{UB}^{-1}$. In order to determine $\mathbf{B}$ and $\mathbf{\Gamma}$ uniquely from the equation $\mathbf{\Gamma} = \mathbf{\Pi B}$ and knowledge of $\mathbf{\Pi}$ (which is in principle observable) conditions on $(\mathbf{B}', \mathbf{\Gamma}')'$ are imposed. For this purpose at least $G-1$ elements in each column of $(\mathbf{B}', \mathbf{\Gamma}')'$ are specified as 0 and one nonzero element in each column is specified (for example, as 1). Then the SEM is *identified*. [Here the notation $\mathbf{B}$ is conventional Cowles Commission for SEM's; see Koopmans (1950).] The specification of 0's implies that if another matrix $(\mathbf{B}^{+'}, \mathbf{\Gamma}^{+'})'$ with the same specification of 0's is related to $(\mathbf{B}, \mathbf{\Gamma})$ by

$$\begin{bmatrix} \mathbf{B}^{+} \\ \mathbf{\Gamma}^{+} \end{bmatrix} = \begin{bmatrix} \mathbf{B} \\ \mathbf{\Gamma} \end{bmatrix} \mathbf{H}, \tag{4.2}$$

where $\mathbf{H}$ is nonsingular of order G, then $\mathbf{H}$ is diagonal. Note that the multiplication of (4.2) corresponds to multiplication of (4.1) on the right by $\mathbf{H}$. If the normalization of the jth component equation of (4.1) is that $\beta_{jj} = 1$, then the specification of 0's and 1's implies $\mathbf{H} = \mathbf{I}_G$.

Suppose that the component equations of (4.1) and the columns of $\mathbf{Y}$ and $\mathbf{Z}$ are numbered so that $(\mathbf{B}', \mathbf{\Gamma}')'$ can be partitioned into G_{I}, G_{E}, K_{i}, and K_{e} rows and n and k columns as

$$\begin{bmatrix} \mathbf{B} \\ \mathbf{\Gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{\mathrm{I}} & \overline{\mathbf{B}}_{\mathrm{I}} \\ \mathbf{0} & \overline{\mathbf{B}}_{\mathrm{E}} \\ \mathbf{\Gamma}_{\mathrm{i}} & \overline{\mathbf{\Gamma}}_{\mathrm{i}} \\ \mathbf{0} & \overline{\mathbf{\Gamma}}_{\mathrm{e}} \end{bmatrix}. \tag{4.3}$$

I and i denote the G_{I} and K_{i} variables *included* in the first n equations, and E and e indicate the G_{E} and K_{e} variables in the model that are *excluded* from the n equations. Here $n \leqslant G_{\mathrm{I}}$, and $\mathbf{B}_{\mathrm{I}}$ has rank n. (Otherwise $\mathbf{B}$ could not be nonsingular.) The *block* of the first n equations [first n columns of $(\mathbf{B}', \mathbf{\Gamma}')'$] is identified if and only if $G_{\mathrm{E}} + K_{\mathrm{e}} \geqslant k = G - n$ and rank $(\overline{\mathbf{B}}_{\mathrm{E}}', \overline{\mathbf{\Gamma}}_{\mathrm{e}}') = k$. The block is *over-identified* if $G_{\mathrm{E}} + K_{\mathrm{e}} > k$. Let

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{nn} & \mathbf{H}_{nk} \\ \mathbf{H}_{kn} & \mathbf{H}_{kk} \end{bmatrix}. \tag{4.4}$$

If $(\mathbf{B}^{+'}, \mathbf{\Gamma}^{+'})'$ has the form of (4.3), then

$$\begin{bmatrix} \mathbf{B}_{\mathrm{I}}^{+} \\ \mathbf{0} \\ \mathbf{\Gamma}_{\mathrm{i}}^{+} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{\mathrm{I}} & \overline{\mathbf{B}}_{\mathrm{I}} \\ \mathbf{0} & \overline{\mathbf{B}}_{\mathrm{E}} \\ \mathbf{\Gamma}_{\mathrm{i}} & \overline{\mathbf{\Gamma}}_{\mathrm{i}} \\ \mathbf{0} & \overline{\mathbf{\Gamma}}_{\mathrm{e}} \end{bmatrix} \begin{bmatrix} \mathbf{H}_{nn} \\ \mathbf{H}_{kn} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{\mathrm{I}}\mathbf{H}_{nn} + \overline{\mathbf{B}}_{\mathrm{I}}\mathbf{H}_{kn} \\ \overline{\mathbf{B}}_{\mathrm{E}}\mathbf{H}_{kn} \\ \mathbf{\Gamma}_{\mathrm{i}}\mathbf{H}_{nn} + \overline{\mathbf{\Gamma}}_{\mathrm{i}}\mathbf{H}_{kn} \\ \overline{\mathbf{\Gamma}}_{\mathrm{e}}\mathbf{H}_{kn} \end{bmatrix}; \tag{4.5}$$

in particular

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{B}}_{\mathrm{E}} \\ \overline{\mathbf{\Gamma}}_{\mathrm{e}} \end{bmatrix} \mathbf{H}_{kn}, \tag{4.6}$$

which implies $\mathbf{H}_{kn} = \mathbf{0}$. Then

$$\begin{bmatrix} \mathbf{B}_{\mathrm{I}}^{+} \\ \mathbf{\Gamma}_{\mathrm{i}}^{+} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{\mathrm{I}} \\ \mathbf{\Gamma}_{\mathrm{i}} \end{bmatrix} \mathbf{H}_{nn}. \tag{4.7}$$

Since $(\mathbf{B}', \mathbf{\Gamma}')$ is identified, $\mathbf{H}_{nn}$ is diagonal.

Let $\mathbf{\Pi}$ $(K \times G)$ be partitioned into K_{i} and K_{e} rows and G_{I} and G_{E} columns as

$$\mathbf{\Pi} = \begin{bmatrix} \mathbf{\Pi}_{\mathrm{iI}} & \mathbf{\Pi}_{\mathrm{iE}} \\ \mathbf{\Pi}_{\mathrm{eI}} & \mathbf{\Pi}_{\mathrm{eE}} \end{bmatrix}. \tag{4.8}$$

Then $\mathbf{\Pi B} = \mathbf{\Gamma}$ is

$$\begin{bmatrix} \mathbf{\Pi}_{\mathrm{iI}} & \mathbf{\Pi}_{\mathrm{iE}} \\ \mathbf{\Pi}_{\mathrm{eI}} & \mathbf{\Pi}_{\mathrm{eE}} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{\mathrm{I}} & \overline{\mathbf{B}}_{\mathrm{I}} \\ \mathbf{0} & \overline{\mathbf{B}}_{\mathrm{E}} \end{bmatrix} = \begin{bmatrix} \mathbf{\Pi}_{\mathrm{iI}}\mathbf{B}_{\mathrm{I}} & \mathbf{\Pi}_{\mathrm{iI}}\overline{\mathbf{B}}_{\mathrm{I}} + \mathbf{\Pi}_{\mathrm{iE}}\overline{\mathbf{B}}_{\mathrm{E}} \\ \mathbf{\Pi}_{\mathrm{eI}}\mathbf{B}_{\mathrm{I}} & \mathbf{\Pi}_{\mathrm{eI}}\overline{\mathbf{B}}_{\mathrm{I}} + \mathbf{\Pi}_{\mathrm{eE}}\overline{\mathbf{B}}_{\mathrm{E}} \end{bmatrix} = \begin{bmatrix} \mathbf{\Gamma}_{\mathrm{i}} & \overline{\mathbf{\Gamma}}_{\mathrm{i}} \\ \mathbf{0} & \overline{\mathbf{\Gamma}}_{\mathrm{e}} \end{bmatrix}. \tag{4.9}$$

The lower left-hand submatrix in (4.9) is

$$\mathbf{\Pi}_{\mathrm{eI}}\mathbf{B}_{\mathrm{I}} = \mathbf{0}. \tag{4.10}$$

If $\mathbf{\Pi}$ is known and rank $\mathbf{\Pi}_{\mathrm{eI}} = G_{\mathrm{I}} - n = k$, then (4.10) can be solved for a matrix $\mathbf{B}_{\mathrm{I}}$ of rank n. Then $\mathbf{\Gamma}$ is determined from the upper left-hand submatrix in (4.9), that is, $\mathbf{\Pi}_{\mathrm{iI}}\mathbf{B}_{\mathrm{I}} = \mathbf{\Gamma}_{\mathrm{i}}$.

This equation (4.10) is of the form of (1.3) with $\mathbf{B}$ replaced by $\mathbf{B}_{\mathrm{I}}$ and $\mathbf{\Pi}$ by $\mathbf{\Pi}_{\mathrm{eI}}$. Any solution of (4.10) can be multiplied on the right by a nonsingular matrix $\mathbf{H}_{nn}$ $(n \times n)$ to give another $\mathbf{B}_{\mathrm{I}}^{+}$ satisfying (4.10) and another $\mathbf{\Gamma}_{\mathrm{i}}^{+} = \mathbf{\Pi}_{\mathrm{iI}}\mathbf{B}_{\mathrm{I}}^{+}$. To eliminate this indeterminateness $n-1$ 0's must be specified in each column of $(\mathbf{B}_{\mathrm{I}}', \mathbf{\Gamma}_{\mathrm{i}}')'$; the submatrix of $(\mathbf{B}_{\mathrm{I}}', \mathbf{\Gamma}_{\mathrm{i}}')'$ obtained by deleting this column and the rows with specified 0's must have rank $n-1$. Then $\mathbf{H}_{nn}$ must be diagonal. We say that each row of $(\mathbf{B}_{\mathrm{I}}', \mathbf{\Gamma}_{\mathrm{i}}')$ is *just-identified within the block* $(\mathbf{B}_{\mathrm{I}}', \mathbf{\Gamma}_{\mathrm{i}}')$.

We now show how the inference in Section 3 can be used for LIML and TSLS estimation of $(\mathbf{B}_{\mathrm{I}}, \mathbf{\Gamma}_{\mathrm{i}})$. The block of equations is

$$\mathbf{Y}_{\mathrm{I}}\mathbf{B}_{\mathrm{I}} = \mathbf{Z}_{\mathrm{i}}\mathbf{\Gamma}_{\mathrm{i}} + \mathbf{U}_{\mathrm{I}}, \tag{4.11}$$

where $\mathbf{Y} = (\mathbf{Y}_\mathrm{I}, \mathbf{Y}_\mathrm{E})$, $\mathbf{Z} = (\mathbf{Z}_\mathrm{i}, \mathbf{Z}_\mathrm{e})$, and $\mathbf{U} = (\mathbf{U}_\mathrm{I}, \mathbf{U}_\mathrm{E})$. The relevant part of the reduced form is

$$\mathbf{Y}_\mathrm{I} = \mathbf{Z}_\mathrm{i}\mathbf{\Pi}_\mathrm{iI} + \mathbf{Z}_\mathrm{e}\mathbf{\Pi}_\mathrm{eI} + \mathbf{V}_\mathrm{I}, \tag{4.12}$$

where $\mathbf{V}_\mathrm{I} = (\mathbf{v}_{\mathrm{I}1}, \ldots, \mathbf{v}_{\mathrm{I}T})'$ is $T \times p$ with $\mathscr{E}\mathbf{v}_{\mathrm{I}t}\mathbf{v}_{\mathrm{I}t}' = \mathbf{\Omega}_\mathrm{II}$. The predetermined variables $\mathbf{Z}$ will be assumed exogenous and nonstochastic. The second-order sample moments of $\mathbf{Y}$ are partitioned as

$$\mathbf{M}_{yy} = \begin{bmatrix} \mathbf{M}_\mathrm{II} & \mathbf{M}_\mathrm{IE} \\ \mathbf{M}_\mathrm{EI} & \mathbf{M}_\mathrm{EE} \end{bmatrix} = \frac{1}{T}\sum_{t=1}^{T}\begin{bmatrix} \mathbf{y}_{\mathrm{I}t}\mathbf{y}_{\mathrm{I}t}' & \mathbf{y}_{\mathrm{I}t}\mathbf{y}_{\mathrm{E}t}' \\ \mathbf{y}_{\mathrm{E}t}\mathbf{y}_{\mathrm{I}t}' & \mathbf{y}_{\mathrm{E}t}\mathbf{y}_{\mathrm{E}t}' \end{bmatrix} = \frac{1}{T}\begin{bmatrix} \mathbf{Y}_\mathrm{I}' \\ \mathbf{Y}_\mathrm{E}' \end{bmatrix}[\mathbf{Y}_\mathrm{I}, \mathbf{Y}_\mathrm{E}]. \tag{4.13}$$

The matrices $\mathbf{M}_{yz}$ and $\mathbf{M}_{zz}$ are partitioned similarly. It is convenient to transform the predetermined variables so that the (block) excluded variables are uncorrelated with the (block) included variables. Let

$$\mathbf{Z}_\perp = \mathbf{Z}_\mathrm{e} - \mathbf{Z}_\mathrm{i}(\mathbf{Z}_\mathrm{i}'\mathbf{Z}_\mathrm{i})^{-1}\mathbf{Z}_\mathrm{i}'\mathbf{Z}_\mathrm{e} \tag{4.14}$$

(the residuals from the regression of the $\mathbf{z}_{\mathrm{e}t}$'s on the $\mathbf{z}_{\mathrm{i}t}$'s). Then the $\mathbf{z}_{\perp t}$'s are uncorrelated with the $\mathbf{z}_{\mathrm{i}t}$'s:

$$\mathbf{M}_{\perp\mathrm{i}} = \frac{1}{T}\mathbf{Z}_\perp'\mathbf{Z}_\mathrm{i} = \mathbf{0}, \quad \mathbf{M}_{\perp\perp} = \frac{1}{T}\mathbf{Z}_\perp'\mathbf{Z}_\perp = \mathbf{M}_\mathrm{ee} - \mathbf{M}_\mathrm{ei}\mathbf{M}_\mathrm{ii}^{-1}\mathbf{M}_\mathrm{ie}. \tag{4.15}$$

The reduced form (4.12) is written as

$$\begin{aligned}\mathbf{Y}_\mathrm{I} &= \mathbf{Z}_\mathrm{i}\mathbf{\Pi}_\mathrm{iI} + [\mathbf{Z}_\perp + \mathbf{Z}_\mathrm{i}(\mathbf{Z}_\mathrm{i}'\mathbf{Z}_\mathrm{i})^{-1}\mathbf{Z}_\mathrm{i}'\mathbf{Z}_\mathrm{e}]\mathbf{\Pi}_\mathrm{eI} + \mathbf{V}_\mathrm{I} \\ &= \mathbf{Z}_\mathrm{i}[\mathbf{\Pi}_\mathrm{iI} + (\mathbf{Z}_\mathrm{i}'\mathbf{Z}_\mathrm{i})^{-1}\mathbf{Z}_\mathrm{i}'\mathbf{Z}_\mathrm{e}\mathbf{\Pi}_\mathrm{ei}] + \mathbf{Z}_\perp\mathbf{\Pi}_\mathrm{eI} + \mathbf{V}_\mathrm{I} \\ &= \mathbf{Z}_\mathrm{i}\overline{\mathbf{\Pi}}_\mathrm{iI} + \mathbf{Z}_\perp\mathbf{\Pi}_\mathrm{eI} + \mathbf{V}_\mathrm{I},\end{aligned} \tag{4.16}$$

where $\overline{\mathbf{\Pi}}_\mathrm{iI} = \mathbf{\Pi}_\mathrm{iI} + \mathbf{M}_\mathrm{ii}^{-1}\mathbf{M}_\mathrm{ie}\mathbf{\Pi}_\mathrm{eI}$. (Note that $\overline{\mathbf{\Pi}}_\mathrm{iI}\mathbf{B}_\mathrm{I} = \mathbf{\Gamma}_\mathrm{i}$ because of (4.10).) The regression of $\mathbf{y}_{\mathrm{I}t}$ on $(\mathbf{z}_{\mathrm{i}t}', \mathbf{z}_{\perp t}')'$ gives the sample regression coefficients

$$\begin{bmatrix} \overline{\mathbf{P}}_\mathrm{iI} \\ \mathbf{P}_\mathrm{eI} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_\mathrm{ii}^{-1}\mathbf{M}_\mathrm{iI} \\ \mathbf{M}_{\perp\perp}^{-1}\mathbf{M}_{\perp\mathrm{I}} \end{bmatrix} = \begin{bmatrix} (\mathbf{Z}_\mathrm{i}'\mathbf{Z}_\mathrm{i})^{-1}\mathbf{Z}_\mathrm{i}'\mathbf{Y}_\mathrm{I} \\ (\mathbf{Z}_\perp'\mathbf{Z}_\perp)^{-1}\mathbf{Z}_\perp'\mathbf{Y}_\mathrm{I} \end{bmatrix}, \tag{4.17}$$

where $\mathbf{M}_{\perp\mathrm{I}} = (1/T)\mathbf{Z}_\perp'\mathbf{Y}_\mathrm{I}$. An estimator of $\mathscr{E}\mathbf{v}_{\mathrm{I}t}\mathbf{v}_{\mathrm{I}t}' = \mathbf{\Omega}_\mathrm{II}$ is

$$\mathbf{W}_\mathrm{II} = \mathbf{M}_\mathrm{II} - \overline{\mathbf{P}}_\mathrm{iI}'\mathbf{M}_\mathrm{ii}\overline{\mathbf{P}}_\mathrm{iI} - \mathbf{P}_\mathrm{eI}'\mathbf{M}_{\perp\perp}\mathbf{P}_\mathrm{eI}. \tag{4.18}$$

Let $\lambda_1 > \cdots > \lambda_{G_\mathrm{I}}$ be the roots of

$$|\mathbf{P}_\mathrm{eI}'\mathbf{M}_{\perp\perp}\mathbf{P}_\mathrm{eI} - \lambda\mathbf{W}_\mathrm{II}| = 0, \tag{4.19}$$

and let $\mathbf{b}_j$ be a solution of

$$(\mathbf{P}_\mathrm{eI}'\mathbf{M}_{\perp\perp}\mathbf{P}_\mathrm{eI} - \lambda_j\mathbf{W}_\mathrm{II})\mathbf{b} = \mathbf{0}, \quad \mathbf{b}'\mathbf{W}_\mathrm{II}\mathbf{b} = 1, \; j = 1, \ldots, G_\mathrm{I}. \tag{4.20}$$

Then

$$\widetilde{\mathbf{B}}_\mathrm{I} = (\mathbf{b}_{k+1}, \ldots, \mathbf{b}_{G_\mathrm{I}}) \tag{4.21}$$

and $\overline{\mathbf{P}}_\mathrm{iI}\widetilde{\mathbf{B}}_\mathrm{I} = \widetilde{\mathbf{\Gamma}}_\mathrm{i}$ form a maximum likelihood estimator of $\mathbf{B}_\mathrm{I}$ and $\mathbf{\Gamma}_\mathrm{i}$. However, a maximum likelihood estimator of $(\mathbf{B}_\mathrm{I}, \mathbf{\Gamma}_\mathrm{i})$ satisfying the 0 conditions is desired.

Suppose the 0 conditions on the jth column are $\beta_{\ell j}=0$ and $\gamma_{mj}=0$ for $n-1$ values of ℓ and m, and suppose that the normalization of the jth column is $\beta_{Lj}=1$ for some L. Then transform $\widetilde{\mathbf{B}}_{\mathrm{I}}$ and $\widetilde{\mathbf{\Gamma}}_{\mathrm{i}}$ by $\mathbf{H}$ the jth column of which is determined by

$$\sum_{g=1}^{n} \widetilde{b}_{Lg}h_{gj}=1, \quad \sum_{g=1}^{n} \widetilde{b}_{\ell g}h_{gj}=0, \quad \sum_{f=1}^{n} \widetilde{\gamma}_{mf}h_{fj}=0 \tag{4.22}$$

for the appropriate ℓ and m.

The block TSLS estimator of $(\mathbf{B}_{\mathrm{I}},\mathbf{\Gamma}_{\mathrm{i}})$ is based on initially estimating $\mathbf{B}_{\mathrm{I}}$ in the form (2.12). Let $\mathbf{P}_{\mathrm{eI}}$ be partitioned into n and k columns $(\mathbf{P}_{\mathrm{e1}},\mathbf{P}_{\mathrm{e2}})$. Then

$$\widetilde{\mathbf{A}}^{\mathrm{TS}}=(\mathbf{P}'_{\mathrm{e2}}\mathbf{M}_{\perp\perp}\mathbf{P}_{\mathrm{e2}})^{-1}\mathbf{P}'_{\mathrm{e2}}\mathbf{M}_{\perp\perp}\mathbf{P}_{\mathrm{e1}} \tag{4.23}$$

is the TSLS estimator of $\mathbf{A}$;

$$\begin{bmatrix} \mathbf{I}_n \\ -\widetilde{\mathbf{A}}^{\mathrm{TS}} \end{bmatrix} \tag{4.24}$$

is a TSLS estimator of $\mathbf{B}_{\mathrm{I}}$; and $\mathbf{P}_{\mathrm{iI}}(\mathbf{I}_n,-\widetilde{\mathbf{A}}^{\mathrm{TS}'})'=\widetilde{\mathbf{\Gamma}}_i^{\mathrm{TS}}$ is a TSLS estimator of $\mathbf{\Gamma}_i$. A block TSLS estimator of $(\mathbf{B}_{\mathrm{I}},\mathbf{\Gamma}_{\mathrm{i}})$ satisfying the identification conditions is obtained from (4.22).

The asymptotic distribution of $\widehat{\mathbf{A}}^{\mathrm{TS}}$ follows from Theorem 1.

Theorem 2. *Under the conditions of Lemma* 1

$$\sqrt{T}\,\mathrm{vec}\,(\widehat{\mathbf{A}}^{\mathrm{TS}}-\mathbf{A})\xrightarrow{d}N[\mathbf{0},\mathbf{\Sigma}_{nn}\otimes(\mathbf{\Pi}'_{\mathrm{e2}}\mathbf{M}^{0}_{\perp\perp}\mathbf{\Pi}_{\mathrm{e2}})^{-1}], \tag{4.25}$$

where $\mathbf{\Sigma}_{nn}=\mathbf{B}'_{\mathrm{I}}\mathbf{\Omega}_{\mathrm{II}}\mathbf{B}_{\mathrm{I}}$.

Corollary 2. *Under the conditions of Lemma* 1 $\sqrt{T}\,\mathrm{vec}\,(\widehat{\mathbf{A}}-\mathbf{A})\xrightarrow{d}N[\mathbf{0},\mathbf{\Sigma}_{nn}\otimes(\mathbf{\Pi}'_{\mathrm{e2}}\mathbf{M}^{00}_{\perp\perp}\mathbf{\Pi}_{\mathrm{e2}})^{-1}]$.

The LIML estimator of $\mathbf{A}$ has the same asymptotic distribution as the TSLS estimator.

Table of Notation

Section2	Section4
$\mathbf{\Pi}$ $(q\times p)$	$\mathbf{\Pi}_{\mathrm{eI}}$ $(K_{\mathrm{e}}\times G_{\mathrm{I}})$
rank $\mathbf{\Pi}=k$	rank $\mathbf{\Pi}_{\mathrm{eI}}=k$
$\mathbf{B}$ $(p\times n)$	$\mathbf{B}_{\mathrm{I}}$ $(G_{\mathrm{I}}\times n)$
rank $\mathbf{B}=n=p-k$	rank $\mathbf{B}_{\mathrm{I}}=n=G_{\mathrm{I}}-k$
$\mathbf{\Pi}\mathbf{B}=\mathbf{0}$	$\mathbf{\Pi}_{\mathrm{eI}}\mathbf{B}_{\mathrm{I}}=\mathbf{0}$

The use of maximum likelihood for reduced rank regression in estimating coefficients in SEM's was suggested by Anderson (1951), and confidence regions for the coefficients were developed. Hannan (1967) repeated some of Anderson's derivation of maximum likelihood estimates. See also Chow et al. (1967).

5. Comparison of estimators

5.1. The just-identification in $\mathbf{B}_\mathrm{I}$

We shall compare estimating the first equation by LIML and estimating it as one of n equations in a block. The block may be over-identified relative to the entire SEM and just-identified within the block.

Suppose the first column of $\mathbf{B}_I$, $\boldsymbol{\beta}_1 = (\beta_{11}, \ldots, \beta_{G_\mathrm{I}1})'$, has $n-1$ specified 0's ($G_\mathrm{I} - n + 1 = G_\mathrm{I}^*$ nonzero elements). Number the columns of $\mathbf{Y}$ so that the first column of $\mathbf{B}_\mathrm{I}$ and $\mathbf{Y}_\mathrm{I}$ are

$$\boldsymbol{\beta}_1 = (1, 0, \ldots, 0, \beta_{n+1,1}, \ldots, \beta_{G_\mathrm{I}1})' = (1, \mathbf{0}, \boldsymbol{\beta}_{1\mathrm{I}}') = (1, \boldsymbol{\beta}_{1\mathrm{E}}', \boldsymbol{\beta}_{1\mathrm{I}}')', \tag{5.1}$$

$$\mathbf{Y}_\mathrm{I} = (\mathbf{y}_1, \mathbf{Y}_{1\mathrm{E}}, \mathbf{Y}_{1\mathrm{I}}), \tag{5.2}$$

respectively. The first column of $\boldsymbol{\Gamma}_i$ (that is, γ_1) has no specified 0. The first equation, $\mathbf{Y}_\mathrm{I}\boldsymbol{\beta}_1 = \mathbf{Z}_\mathrm{i}\gamma_1 + \mathbf{u}_1$, is

$$\mathbf{y}_1 + \mathbf{Y}_{1\mathrm{I}}\boldsymbol{\beta}_{1\mathrm{I}} = \mathbf{Z}_\mathrm{i}\gamma_1 + \mathbf{u}_1. \tag{5.3}$$

Partition $\mathbf{P}_{\mathrm{eI}}$ into 1, $n-1$, and $G_\mathrm{I} - n$ columns as $\mathbf{P}_{\mathrm{eI}} = (\mathbf{p}_{\mathrm{e}1}, \mathbf{P}_{\mathrm{e}1\mathrm{E}}, \mathbf{P}_{\mathrm{e}1\mathrm{I}})$ and $\mathbf{W}_{\mathrm{II}}$ into 1, $n-1$, and $G_\mathrm{I} - n$ rows and columns

$$\mathbf{W}_{\mathrm{II}} = \begin{bmatrix} w_{11} & \mathbf{w}_{1,1\mathrm{E}} & \mathbf{w}_{1,1\mathrm{I}} \\ \mathbf{w}_{1\mathrm{E},1} & \mathbf{W}_{1\mathrm{E},1\mathrm{E}} & \mathbf{W}_{1\mathrm{E},1\mathrm{I}} \\ \mathbf{W}_{1\mathrm{I},1} & \mathbf{w}_{1\mathrm{I},1\mathrm{E}} & \mathbf{W}_{1\mathrm{I},1\mathrm{I}} \end{bmatrix}. \tag{5.4}$$

To define the single-equation LIML estimator of $\boldsymbol{\beta}_1$ let $\lambda_1 > \cdots > \lambda_{G_\mathrm{I}^*}$ be the roots of

$$\left| \begin{pmatrix} \mathbf{p}_{\mathrm{e}1}' \\ \mathbf{P}_{\mathrm{e}1\mathrm{I}}' \end{pmatrix} \mathbf{M}_{\perp\perp}(\mathbf{p}_{\mathrm{e}1}, \mathbf{P}_{\mathrm{e}1\mathrm{I}}) - \lambda \begin{pmatrix} w_{11} & \mathbf{w}_{1,1\mathrm{I}} \\ \mathbf{w}_{1\mathrm{I},1} & \mathbf{W}_{1\mathrm{I},1\mathrm{I}} \end{pmatrix} \right| = 0. \tag{5.5}$$

Let $\mathbf{b}_j$ be the solution to

$$\left[\begin{pmatrix} \mathbf{p}_{\mathrm{e}1}' \\ \mathbf{P}_{\mathrm{e}1\mathrm{I}}' \end{pmatrix} \mathbf{M}_{\perp\perp}(\mathbf{p}_{\mathrm{e}1}, \mathbf{P}_{\mathrm{e}1\mathrm{I}}) - \lambda_j \begin{pmatrix} w_{11} & \mathbf{w}_{1,1\mathrm{I}} \\ \mathbf{w}_{1\mathrm{I},1} & \mathbf{W}_{1\mathrm{I},1\mathrm{I}} \end{pmatrix} \right] \mathbf{b} = \mathbf{0} \tag{5.6}$$

and

$$\mathbf{b}' \begin{pmatrix} w_{11} & \mathbf{w}_{1,1\mathrm{I}} \\ \mathbf{w}_{1\mathrm{I},1} & \mathbf{W}_{1\mathrm{I},1\mathrm{I}} \end{pmatrix} \mathbf{b} = 1, \tag{5.7}$$

$j = 1, \ldots, G_\mathrm{I}^*$. Write $\mathbf{b}_{G_\mathrm{I}^*} = (b_1, \mathbf{b}_{1\mathrm{I}}')'$. Then the single-equation LIML estimator of $\boldsymbol{\beta}_{1\mathrm{I}}$ is $(1/b_1)\mathbf{b}_{1\mathrm{I}}$. The estimator of γ_i is $(1/b_1)(\mathbf{p}_{\mathrm{i}1}, \mathbf{P}_{\mathrm{i}1\mathrm{I}})\mathbf{b}_{G_\mathrm{I}^*}$.

On the other hand, a LIML estimator of the block without the just-identification is $\widetilde{\mathbf{B}}_{\mathrm{I}}$, defined by (4.21). To calculate the estimate of the coefficients of the first identified equation in the block find $\mathbf{h}_1 = (h_{11}, \ldots, h_{n1})'$ from

$$\sum_{g=1}^{n} \tilde{b}_{1g} h_{g1} = 1, \quad \sum_{g=1}^{n} \tilde{b}_{\ell g} h_{g1} = 0, \quad \ell = 2, \ldots, n. \tag{5.8}$$

If $\widetilde{\mathbf{B}}_{\mathrm{I}} = (\widetilde{\mathbf{B}}_1', \widetilde{\mathbf{B}}_2')'$, where $\widetilde{\mathbf{B}}_1$ is $n \times n$, then $\mathbf{h}_1$ is the first column of $\widetilde{\mathbf{B}}_1^{-1}$. Note that the first column of $\widetilde{\mathbf{B}}_{\mathrm{I}}$ without the specified 0's is different from $\mathbf{b}_{G_{\mathrm{I}}^*}$, the single-equation LIML estimator. See Appendix A.

A single-equation TSLS estimator of $\boldsymbol{\beta}_{1\mathrm{I}}$ is

$$-(\mathbf{P}_{\mathrm{e1I}}'\mathbf{M}_{\perp\perp}\mathbf{P}_{\mathrm{e1I}})^{-1}\mathbf{P}_{\mathrm{e1I}}'\mathbf{M}_{\perp\perp}\mathbf{p}_{\mathrm{e1}}. \tag{5.9}$$

A block-equation TSLS estimator of $\mathbf{B}_{\mathrm{I}}$ in the form of (3.17) is $(\mathbf{I}, -\widehat{\mathbf{A}}^{\mathrm{TS}'})'$ where

$$\widehat{\mathbf{A}}^{\mathrm{TS}} = (\mathbf{P}_{\mathrm{e1I}}'\mathbf{M}_{\perp\perp}\mathbf{P}_{\mathrm{e1I}})^{-1}\mathbf{P}_{\mathrm{1Ie}}'\mathbf{M}_{\perp\perp}\mathbf{P}_{\mathrm{eI}}. \tag{5.10}$$

The single-equation TSLS estimator (5.9) is the first column of the block-equation TSLS estimator (5.10). The notation here was selected so that the first n components of $\boldsymbol{\beta}_1$ agreed with the first n components in the first column of $(\mathbf{I}_n, -\mathbf{A}')'$. This correspondence facilitates comparison of the single-equation TSLS estimator with the block-equation TSLS estimator. However, even in this situation, the single-equation LIML estimator may be different from the block-equation LIML estimator.

5.2. *The just-identification in* $\boldsymbol{\Gamma}_{\mathrm{i}}$

Suppose the first column of $\boldsymbol{\Gamma}_{\mathrm{i}}, \boldsymbol{\gamma}_1 = (\gamma_{11}, \ldots, \gamma_{K_{\mathrm{i}}1})'$, has $n-1$ specified 0's. Number the columns of $\mathbf{Z}$ so that the first column of $\boldsymbol{\Gamma}_{\mathrm{i}}$ and $\mathbf{Z}_{\mathrm{i}}$ are

$$\boldsymbol{\gamma}_1 = (\gamma_{11}, \ldots, \gamma_{K_{\mathrm{i}}^*1}, 0, \ldots, 0)' = (\boldsymbol{\gamma}_{1\mathrm{i}}', \mathbf{0}) = (\boldsymbol{\gamma}_{1\mathrm{i}}', \boldsymbol{\gamma}_{1\mathrm{e}}')', \tag{5.11}$$

$$\mathbf{Z}_{\mathrm{i}} = (\mathbf{Z}_{1\mathrm{i}}, \mathbf{Z}_{1\mathrm{e}}), \tag{5.12}$$

where $K_{\mathrm{i}}^* = K_{\mathrm{i}} - n + 1$. Number the columns of $\mathbf{Y}$ so that $\beta_{11} = 1$; none of the other $G_{\mathrm{I}} - 1$ components of $\boldsymbol{\beta}_1$ is specified 0. The first equation, $\mathbf{Y}_{\mathrm{I}}\boldsymbol{\beta}_1 = \mathbf{Z}_{\mathrm{i}}\boldsymbol{\gamma}_1 + \mathbf{u}_1$, is

$$\mathbf{y}_1 + \mathbf{Y}_{1\mathrm{I}}\boldsymbol{\beta}_{1\mathrm{I}} = \mathbf{Z}_{1\mathrm{i}}\boldsymbol{\gamma}_{1\mathrm{i}} + \mathbf{u}_1, \tag{5.13}$$

where $\boldsymbol{\beta}_1 = (1, \boldsymbol{\beta}_{1\mathrm{I}}')'$ and $\mathbf{Y}_{\mathrm{I}} = (\mathbf{y}_1, \mathbf{Y}_{1\mathrm{I}})$. The exogenous variables excluded from the first equation are $(\mathbf{Z}_{1\mathrm{e}}, \mathbf{Z}_{\mathrm{e}})$. The corresponding part of the reduced form is

$$\mathbf{Y}_{\mathrm{I}} = \mathbf{Z}_{1\mathrm{i}}\boldsymbol{\Pi}_{1\mathrm{i,I}} + (\mathbf{Z}_{1\mathrm{e}}, \mathbf{Z}_{\mathrm{e}})\begin{pmatrix}\boldsymbol{\Pi}_{1\mathrm{e,I}} \\ \boldsymbol{\Pi}_{\mathrm{eI}}\end{pmatrix} + \mathbf{V}_{\mathrm{I}}. \tag{5.14}$$

Let $\mathbf{\Pi}$ be partitioned into $K_{1\mathrm{i}}$, $K_{1\mathrm{e}}$, and K_{e} rows and G_{I} and G_{E} columns

$$\mathbf{\Pi} = \begin{bmatrix} \mathbf{\Pi}_{1\mathrm{i},\mathrm{I}} & \mathbf{\Pi}_{1\mathrm{i},\mathrm{E}} \\ \mathbf{\Pi}_{1\mathrm{e},\mathrm{I}} & \mathbf{\Pi}_{1\mathrm{e},\mathrm{E}} \\ \mathbf{\Pi}_{\mathrm{eI}} & \mathbf{\Pi}_{\mathrm{eE}} \end{bmatrix}. \tag{5.15}$$

Then

$$\begin{bmatrix} \mathbf{\Pi}_{1\mathrm{e},\mathrm{I}} \\ \mathbf{\Pi}_{\mathrm{eI}} \end{bmatrix} \boldsymbol{\beta}_1 = \mathbf{0}. \tag{5.16}$$

Let

$$\mathbf{Z}_{\perp}^{*} = (\mathbf{Z}_{1\mathrm{e}}, \mathbf{Z}_{\mathrm{e}}) - \mathbf{Z}_{1\mathrm{i}}(\mathbf{Z}_{1\mathrm{i}}'\mathbf{Z}_{1\mathrm{i}})^{-1}\mathbf{Z}_{1\mathrm{i}}'(\mathbf{Z}_{1\mathrm{e}}, \mathbf{Z}_{\mathrm{e}}), \tag{5.17}$$

$$\overline{\mathbf{\Pi}}_{1\mathrm{i},\mathrm{I}} = \mathbf{\Pi}_{1\mathrm{i},\mathrm{I}} + (\mathbf{Z}_{1\mathrm{i}}'\mathbf{Z}_{1\mathrm{i}})^{-1}\mathbf{Z}_{1\mathrm{i}}'(\mathbf{Z}_{1\mathrm{e}}, \mathbf{Z}_{\mathrm{e}}) \begin{pmatrix} \mathbf{\Pi}_{1\mathrm{e},\mathrm{I}} \\ \mathbf{\Pi}_{\mathrm{eI}} \end{pmatrix}. \tag{5.18}$$

Then the reduced form (5.14) is

$$\mathbf{Y}_{\mathrm{I}} = \mathbf{Z}_{1\mathrm{i}}\overline{\mathbf{\Pi}}_{1\mathrm{i},\mathrm{I}} + \mathbf{Z}_{\perp}^{*} \begin{pmatrix} \mathbf{\Pi}_{1\mathrm{e},\mathrm{I}} \\ \mathbf{\Pi}_{\mathrm{eI}} \end{pmatrix} + \mathbf{V}_{\mathrm{I}}, \tag{5.19}$$

and $\mathbf{Z}_{1\mathrm{i}}'\mathbf{Z}_{\perp}^{*} = \mathbf{0}$. Define

$$\begin{bmatrix} \overline{\mathbf{P}}_{1\mathrm{i},\mathrm{I}} \\ \begin{pmatrix} \mathbf{P}_{1\mathrm{e},\mathrm{I}} \\ \mathbf{P}_{\mathrm{eI}} \end{pmatrix} \end{bmatrix} = \begin{bmatrix} (\mathbf{Z}_{1\mathrm{i}}'\mathbf{Z}_{1\mathrm{i}})^{-1}\mathbf{Z}_{1\mathrm{i}}'\mathbf{Y}_{\mathrm{I}} \\ (\mathbf{Z}_{\perp}^{*\prime}\mathbf{Z}_{\perp}^{*})^{-1}\mathbf{Z}_{\perp}^{*\prime}\mathbf{Y}_{\mathrm{I}} \end{bmatrix}. \tag{5.20}$$

Another form of the estimator $\mathbf{W}_{\mathrm{II}}$ is

$$\mathbf{W}_{\mathrm{II}} = \mathbf{M}_{\mathrm{II}} - \overline{\mathbf{P}}_{1\mathrm{i},\mathrm{I}}' \frac{1}{T} \mathbf{Z}_{1\mathrm{i}}'\mathbf{Z}_{1\mathrm{i}}\overline{\mathbf{P}}_{1\mathrm{i},\mathrm{I}} - (\mathbf{P}_{1\mathrm{e},\mathrm{I}}', \mathbf{P}_{\mathrm{eI}}')\frac{1}{T} \mathbf{Z}_{\perp}^{*\prime}\mathbf{Z}_{\perp}^{*} \begin{bmatrix} \mathbf{P}_{1\mathrm{e},\mathrm{I}} \\ \mathbf{P}_{\mathrm{eI}} \end{bmatrix}. \tag{5.21}$$

Let $\lambda_1 > \cdots > \lambda_{G_{\mathrm{I}}}$ be the roots of

$$\left| (\mathbf{P}_{1\mathrm{e},\mathrm{I}}', \mathbf{P}_{\mathrm{eI}}')\frac{1}{T} \mathbf{Z}_{\perp}^{*\prime}\mathbf{Z}_{\perp}^{*} \begin{pmatrix} \mathbf{P}_{1\mathrm{e},\mathrm{I}} \\ \mathbf{P}_{\mathrm{eI}} \end{pmatrix} - \lambda \mathbf{W}_{\mathrm{II}} \right| = 0. \tag{5.22}$$

Let $\mathbf{b}_j$ be the solution to

$$\left[(\mathbf{P}_{1\mathrm{e},\mathrm{I}}', \mathbf{P}_{\mathrm{eI}}')\frac{1}{T} \mathbf{Z}_{\perp}^{*\prime}\mathbf{Z}_{\perp}^{*} \begin{pmatrix} \mathbf{P}_{1\mathrm{e},\mathrm{I}} \\ \mathbf{P}_{\mathrm{e},\mathrm{I}} \end{pmatrix} - \lambda_j \mathbf{W}_{\mathrm{II}} \right] \mathbf{b} = \mathbf{0} \tag{5.23}$$

and $\mathbf{b}'\mathbf{W}_{\mathrm{II}}\mathbf{b} = 1$, $j = 1, \ldots, G_{\mathrm{I}}$. Write $\mathbf{b}_{G_{\mathrm{I}}} = (b_1, \mathbf{b}_{1\mathrm{I}}')'$. The single-equation LIML estimator of $\boldsymbol{\beta}_1$ is $(1/b_1)\mathbf{b}_{G_{\mathrm{I}}}$.

The block LIML estimator of $\mathbf{B}_{\mathrm{I}}$ without the just-identification is $\widetilde{\mathbf{B}}_{\mathrm{I}}$ defined by (4.21). To calculate the just-identified equation in the block find

$\mathbf{h}_1 = (h_{11}, \ldots, h_{n1})'$ from

$$\sum_{g=1}^{n} \widetilde{b}_{1g} h_{g1} = 1, \quad \sum_{g=1}^{n} \widetilde{\gamma}_{\ell g} h_{g1} = 0, \quad \ell = 1, \ldots, n-1. \tag{5.24}$$

The linear combination $\widetilde{\mathbf{B}}_{\mathrm{I}}\mathbf{h}_1$ does not satisfy (5.23); that is, the identified block LIML estimator of $\boldsymbol{\beta}_{1\mathrm{I}}$ is not the single-equation LIML estimator.

5.3. An instrumental variables approach

Write the block of equations (4.11) as

$$\mathbf{Y}_{\mathrm{I}}\mathbf{B}_{\mathrm{I}} - \mathbf{Z}_{\mathrm{i}}\boldsymbol{\Gamma}_{\mathrm{i}} = \mathbf{U}_{\mathrm{I}}. \tag{5.25}$$

Then

$$(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Y}_{\mathrm{I}}, \mathbf{Z}_{\mathrm{i}}) \begin{bmatrix} \mathbf{B}_{\mathrm{I}} \\ -\boldsymbol{\Gamma}_{\mathrm{i}} \end{bmatrix} = \mathbf{M}_{zz}^{-1}(\mathbf{M}_{z\mathrm{I}}\mathbf{B}_{\mathrm{I}} - \mathbf{M}_{z\mathrm{i}}\boldsymbol{\Gamma}_{\mathrm{i}}) = \mathbf{M}_{zz}^{-1}\frac{1}{T}\mathbf{Z}'\mathbf{U}_{\mathrm{I}} \xrightarrow{p} \mathbf{0}. \tag{5.26}$$

Since

$$\mathbf{M}_{zz}^{-1}\mathbf{M}_{z\mathrm{I}} = \begin{bmatrix} \mathbf{P}_{\mathrm{iI}} \\ \mathbf{P}_{\mathrm{eI}} \end{bmatrix} \xrightarrow{p} \begin{bmatrix} \boldsymbol{\Pi}_{\mathrm{iI}} \\ \boldsymbol{\Pi}_{\mathrm{eI}} \end{bmatrix}, \tag{5.27}$$

$$\mathbf{M}_{zz}^{-1}\mathbf{M}_{z\mathrm{i}} = \begin{bmatrix} \mathbf{I}_{K_{\mathrm{i}}} \\ \mathbf{0} \end{bmatrix}, \tag{5.28}$$

the probability limit of (5.26) is

$$\begin{bmatrix} \boldsymbol{\Pi}_{\mathrm{iI}} & \mathbf{I}_{K_{\mathrm{i}}} \\ \boldsymbol{\Pi}_{\mathrm{eI}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{\mathrm{I}} \\ -\boldsymbol{\Gamma}_{\mathrm{i}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Pi}_{\mathrm{iI}}\mathbf{B}_{\mathrm{I}} - \boldsymbol{\Gamma}_{\mathrm{i}} \\ \boldsymbol{\Pi}_{\mathrm{eI}}\mathbf{B}_{\mathrm{I}} \end{bmatrix} = \mathbf{0}. \tag{5.29}$$

If $(\boldsymbol{\Pi}_{\mathrm{iI}}', \boldsymbol{\Pi}_{\mathrm{eI}}')$ is given, (5.29) can be solved for $\mathbf{B}_{\mathrm{I}}$ and $\boldsymbol{\Gamma}_{\mathrm{i}}$, but there is the indeterminacy of multiplication of the solution on the right by a nonsingular $n \times n$ matrix. The indeterminacy can be eliminated by requiring that the $n \times n$ matrix consisting of some n rows of $(\mathbf{B}_{\mathrm{I}}', -\boldsymbol{\Gamma}_{\mathrm{i}}')'$ constitute $\mathbf{I}_n$. The requirement is a special case of eliminating the indeterminacy by requiring the just-identification of n equations as specified in Section 4.

A consistent estimator of $(\mathbf{B}_{\mathrm{I}}', \boldsymbol{\Gamma}_{\mathrm{i}}')'$ is obtained by a two-stage procedure, first setting to $\mathbf{0}$ some $\mathbf{G}_{\mathrm{I}} + K_{\mathrm{i}} - n$ rows of

$$\mathbf{M}_{zz}^{-1}[\mathbf{M}_{z\mathrm{I}}, \mathbf{M}_{z\mathrm{i}}] \begin{bmatrix} \widehat{\mathbf{B}}_{\mathrm{I}} \\ -\widehat{\boldsymbol{\Gamma}}_{\mathrm{i}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{\mathrm{iI}} & \mathbf{I}_{K_{\mathrm{i}}} \\ \mathbf{P}_{\mathrm{eI}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{B}}_{\mathrm{I}} \\ -\widehat{\boldsymbol{\Gamma}}_{\mathrm{i}} \end{bmatrix}. \tag{5.30}$$

If that set of equations is the last $G_{\mathrm{I}} - n + k_i$ equations and $\widehat{\mathbf{B}} = (\mathbf{I}_n, -\widehat{\mathbf{A}}')'$, the equations are

$$\mathbf{0} = \begin{bmatrix} \mathbf{P}'_{\mathrm{i}2} & \mathbf{P}'_{\mathrm{e}2} \\ \mathbf{I}_{K_{\mathrm{i}}} & \mathbf{0} \end{bmatrix} \mathbf{M}_{zz} \begin{bmatrix} \mathbf{P}_{\mathrm{i}1} & \mathbf{P}_{\mathrm{i}2} & \mathbf{I}_{K_{\mathrm{i}}} \\ \mathbf{P}_{\mathrm{e}1} & \mathbf{P}_{\mathrm{e}2} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n \\ -\widehat{\mathbf{A}} \\ -\widehat{\mathbf{\Gamma}}_{\mathrm{i}} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{P}'_{\mathrm{i}2} & \mathbf{P}'_{\mathrm{e}2} \\ \mathbf{I}_{K_{\mathrm{i}}} & \mathbf{0} \end{bmatrix} \mathbf{M}_{zz} \begin{bmatrix} \mathbf{P}_{\mathrm{i}1} \\ \mathbf{P}_{\mathrm{e}1} \end{bmatrix} - \begin{bmatrix} \mathbf{P}'_{\mathrm{i}2} & \mathbf{P}'_{\mathrm{e}2} \\ \mathbf{I}_{K_{\mathrm{i}}} & \mathbf{0} \end{bmatrix} \mathbf{M}_{zz} \begin{bmatrix} \mathbf{P}_{\mathrm{i}2} & \mathbf{I}_{K_{\mathrm{i}}} \\ \mathbf{P}_{\mathrm{e}2} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{A}} \\ \widehat{\mathbf{\Gamma}}'_{\mathrm{i}} \end{bmatrix}. \tag{5.31}$$

The block TSLS estimator of $\mathbf{A}$ and $\mathbf{\Gamma}_{\mathrm{i}}$ is

$$\begin{bmatrix} \widehat{\mathbf{A}}^{\mathrm{T}} \\ \widehat{\mathbf{\Gamma}}_{\mathrm{i}}^{\mathrm{T}} \end{bmatrix} = \left\{ \begin{bmatrix} \mathbf{P}'_{\mathrm{i}2} & \mathbf{P}'_{\mathrm{e}2} \\ \mathbf{I}_{K_{\mathrm{i}}} & \mathbf{0} \end{bmatrix} \mathbf{M}_{zz} \begin{bmatrix} \mathbf{P}_{\mathrm{i}2} & \mathbf{I}_{K_{\mathrm{i}}} \\ \mathbf{P}_{\mathrm{e}2} & \mathbf{0} \end{bmatrix} \right\}^{-1} \begin{bmatrix} \mathbf{P}'_{\mathrm{i}2} & \mathbf{P}'_{\mathrm{e}2} \\ \mathbf{I}_{K_{\mathrm{i}}} & \mathbf{0} \end{bmatrix} \mathbf{M}_{zz} \begin{bmatrix} \mathbf{P}_{\mathrm{i}1} \\ \mathbf{P}_{\mathrm{e}1} \end{bmatrix}. \tag{5.32}$$

Suppose the normalization condition for the jth column of $(\mathbf{B}'_{\mathrm{I}}, \mathbf{\Gamma}'_{\mathrm{i}})'$ is $\beta_{Lj} = 1$ for a particular L and the jth row is just-identified by $\beta_{\ell j} = 0$ and $\gamma_{mj} = 0$ for $n-1$ specified indices ℓ and m, then the jth column of the transformation matrix $\mathbf{H}_{nn}$ is the solution of (4.22).

Estimator (5.32) of $\mathbf{A}$ and $\mathbf{\Gamma}_{\mathrm{i}}$ is identical to (4.23) and $\mathbf{P}_{\mathrm{iI}}(\mathbf{I}_n, -\widehat{\mathbf{A}}^{\mathrm{TS}'})' = \widehat{\mathbf{\Gamma}}_{\mathrm{i}}^{\mathrm{TS}}$. From (5.32) we derive

$$\sqrt{T}\,\mathrm{vec} \begin{bmatrix} \widehat{\mathbf{A}}^{\mathrm{TS}} - \mathbf{A} \\ \widehat{\mathbf{\Gamma}}_{\mathrm{i}}^{\mathrm{TS}} - \mathbf{\Gamma}_{\mathrm{i}} \end{bmatrix} \xrightarrow{d} N\left\{ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \mathbf{\Sigma}_{nn} \otimes \left[\begin{pmatrix} \mathbf{\Pi}'_{\mathrm{i}2} & \mathbf{\Pi}'_{\mathrm{e}2} \\ \mathbf{I}_{K_{\mathrm{i}}} & \mathbf{0} \end{pmatrix} \mathbf{M}^0_{zz} \begin{pmatrix} \mathbf{\Pi}_{\mathrm{i}2} & \mathbf{I}_{K_{\mathrm{i}}} \\ \mathbf{\Pi}_{\mathrm{e}2} & \mathbf{0} \end{pmatrix} \right]^{-1} \right\}. \tag{5.33}$$

5.4. *Just-identification in both* $\mathbf{B}_{\mathrm{I}}$ *and* $\mathbf{\Gamma}_{\mathrm{i}}$

Now consider just-identification that involves both $\mathbf{B}_{\mathrm{I}}$ and $\mathbf{\Gamma}_{\mathrm{i}}$. Suppose the components of $\mathbf{y}_{\mathrm{I}t}$ and $\mathbf{z}_{\mathrm{i}t}$ are numbered so that the first row of $(\mathbf{B}'_{\mathrm{I}}, \mathbf{\Gamma}'_{\mathrm{i}})$ is

$$(\boldsymbol{\beta}'_1, \boldsymbol{\gamma}'_1) = \Bigg(\underbrace{\overbrace{\beta_{11}, \ldots, \beta_{G^*_{\mathrm{I}}-1,1}}^{G^*_{\mathrm{I}}-1}, \overset{1}{1}\, \overbrace{0, \ldots, 0}^{m}}_{G_{\mathrm{I}}}, \quad \underbrace{\overbrace{0, \ldots, 0}^{n-m-1}, \overbrace{\gamma_{n-m,1}, \ldots, \gamma_{1K_{\mathrm{i},1}}}^{K^*_{\mathrm{i}}}}_{K_{\mathrm{i}}} \Bigg)$$
$$= (\boldsymbol{\beta}'_{1\mathrm{I}}, 1, \mathbf{0}, \mathbf{0}, \boldsymbol{\gamma}'_{1\mathrm{i}}), \tag{5.34}$$

where $G^*_{\mathrm{I}} = G_{\mathrm{I}} - m$ and $K^*_{\mathrm{i}} = K_{\mathrm{i}} - (n - m - 1)$ are the numbers of endogenous and exogenous variables in the first equation. The number of specified 0's in $\boldsymbol{\beta}_1$ and $\boldsymbol{\gamma}_1$ is $n-1$.

The single-equation TSLS estimator of the coefficients of the first equation is based on the following:

Included y_t's : $(y_{1t}, \ldots, y_{G^*_{\mathrm{I}}t}) = (\mathbf{y}'_{1\mathrm{I},t}, y_{G^*_{\mathrm{I}}t})$,

Excluded y_t's : $(y_{G^*_{\mathrm{I}}+1,t}, \ldots, y_{Gt}) = (\mathbf{y}'_{1\mathrm{E},t}, \mathbf{y}'_{\mathrm{E},t})$,

$$\text{Included } z_t\text{'s}: (z_{n-m,t}, \ldots, z_{K_i t}) = \mathbf{z}'_{1i,t},$$
$$\text{Excluded } z_t\text{'s}: (z_{1t}, \ldots, z_{n-m-1,t}, z_{K_i+1,t}, \ldots, z_{Kt}) = (\mathbf{z}'_{1e,t}, \mathbf{z}'_{et}). \tag{5.35}$$

Note that $(\mathbf{z}'_{1e,t}, \mathbf{z}'_{et})$ includes z_t's included in the block but excluded from the first equation. The first equation is

$$\mathbf{y}_{G_I^*} = -\mathbf{Y}_{1I}\boldsymbol{\beta}_{1I} + \mathbf{Z}_{1i}\boldsymbol{\gamma}_{1i} + \mathbf{u}_{1\cdot}. \tag{5.36}$$

The single-equation TSLS estimator of $(\boldsymbol{\beta}'_{1I}, \boldsymbol{\gamma}'_{1i})$ is

$$\left\{ \begin{bmatrix} -\mathbf{Y}'_{1I}\mathbf{Z} \\ \mathbf{Z}'_{1i}\mathbf{Z} \end{bmatrix} (\mathbf{Z}'\mathbf{Z})^{-1}(-\mathbf{Z}'\mathbf{Y}_{1I}, \mathbf{Z}'\mathbf{Z}_{1i}) \right\}^{-1} \begin{bmatrix} -\mathbf{Y}'_{1I}\mathbf{Z} \\ \mathbf{Z}'_{1i}\mathbf{Z} \end{bmatrix} (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}_m. \tag{5.37}$$

Now we turn to the block TSLS estimator of $\mathbf{B}_I$ and $\boldsymbol{\Gamma}_i$. Let

$$\begin{bmatrix} \mathbf{B}_I \\ \boldsymbol{\Gamma}_i \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{1I} \\ \mathbf{C} \\ \boldsymbol{\Gamma}_{1i} \end{bmatrix}, \tag{5.38}$$

where $\mathbf{B}_{1I}$ has $G_I^* - 1$ rows, $\mathbf{C}$ has n rows, and $\boldsymbol{\Gamma}_{1i}$ has K_i^* rows. Define

$$\begin{bmatrix} \mathbf{B}^*_{1I} \\ \mathbf{I}_n \\ \boldsymbol{\Gamma}^*_{1i} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_I \\ \boldsymbol{\Gamma}_i \end{bmatrix} \mathbf{C}^{-1} = \begin{bmatrix} \mathbf{B}_{1I}\mathbf{C}^{-1} \\ \mathbf{I}_n \\ \boldsymbol{\Gamma}_{1i}\mathbf{C}^{-1} \end{bmatrix}. \tag{5.39}$$

The block of equations can be written

$$(\mathbf{y}_{G_I^*}, \mathbf{Y}_{1E}, -\mathbf{Z}_{1e}) = (-\mathbf{Y}_{1I}, \mathbf{Z}_{1i}) \begin{bmatrix} \mathbf{B}^*_{1I} \\ \boldsymbol{\Gamma}^*_{1i} \end{bmatrix} + \mathbf{U}_I^*, \tag{5.40}$$

where $\mathbf{U}_I^* = \mathbf{U}_I\mathbf{C}^{-1}$.

In place of (5.31) we have

$$\begin{bmatrix} \mathbf{M}_{1Iz} \\ -\mathbf{M}_{1iz} \end{bmatrix} \mathbf{M}_{zz}^{-1}[\mathbf{M}_{z1I}, (\mathbf{M}_{zG_I^*}, \mathbf{M}_{z1E}, -\mathbf{M}_{z1e}), -\mathbf{M}_{z1i}] \begin{bmatrix} \mathbf{B}^*_{1I} \\ \mathbf{I}_n \\ \boldsymbol{\Gamma}^*_{1i} \end{bmatrix} = \mathbf{0}. \tag{5.41}$$

The TSLS estimator of $(\mathbf{B}^{*'}_{1I}, \boldsymbol{\Gamma}^{*'}_{1i})'$ is

$$\begin{bmatrix} \widehat{\mathbf{B}}^*_I \\ \widehat{\boldsymbol{\Gamma}}^*_i \end{bmatrix} = \left\{ \begin{bmatrix} \mathbf{M}_{1Iz} \\ \mathbf{M}_{1iz} \end{bmatrix} \mathbf{M}_{zz}^{-1}(\mathbf{M}_{z1I}, -\mathbf{M}_{z1i}) \right\}^{-1} \begin{bmatrix} \mathbf{M}_{1Iz} \\ \mathbf{M}_{1iz} \end{bmatrix} \mathbf{M}_{zz}^{-1}(-\mathbf{M}_{zG_I^*}, -\mathbf{M}_{z1E}, \mathbf{M}_{z1e}). \tag{5.42}$$

The single-equation TSLS estimator of the coefficients of the first component equation is the first column of the block TSLS estimator.

As pointed out in Section 5.3 the set of block TSLS estimators is determined by the selection of $G_I + K_i - n$ rows of (5.30) set to $\mathbf{0}$. The indeterminacy of multiplication on the right by a nonsingular $n \times n$ matrix can be removed to requiring some n rows of $(\mathbf{B}_{\mathrm{I}}', \mathbf{\Gamma}_{\mathrm{i}}')'$ to constitute $\mathbf{I}_n$. Here those rows correspond to the 1 and 0's in $(\boldsymbol{\beta}_1', \gamma_1')'$.

6. Some examples

Example 1. Consider the model for $G = 3$ and $K = 3$

$$\begin{bmatrix} \mathbf{B} \\ \mathbf{\Gamma} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \phi & \psi & 1 \\ & & \\ 0 & 0 & \delta \\ 0 & 0 & \tau \\ 0 & 0 & \theta \end{bmatrix}, \tag{6.1}$$

$\mathbf{y}_t = (y_{1t}, y_{2t}, y_{3t})'$, $\mathbf{z}_t = (z_{1t}, z_{2t}, z_{3t})'$. The first equation is

$$\mathbf{y}_1 + \mathbf{y}_3\phi = \mathbf{u}_1. \tag{6.2}$$

Then if $n = 1$ $\boldsymbol{\beta}_{\mathrm{I}} = (1, \phi)'$, $\boldsymbol{\beta}_{\mathrm{E}} = 0$, $\gamma_{\mathrm{e}} = (0, 0, 0)'$, and γ_{i} is vacuous. The relevant part of $\mathbf{P}\,(3 \times 3)$ is

$$\begin{bmatrix} p_{11} & p_{13} \\ p_{21} & p_{23} \\ p_{31} & p_{33} \end{bmatrix} = (\mathbf{p}_{\mathrm{e}1}, \mathbf{p}_{\mathrm{e}3}); \tag{6.3}$$

the TSLS estimator of ϕ is

$$-\frac{\mathbf{p}_{\mathrm{e}3}'\mathbf{M}_{zz}\mathbf{p}_{\mathrm{e}1}}{\mathbf{p}_{\mathrm{e}3}'\mathbf{M}_{zz}\mathbf{p}_{\mathrm{e}3}'}; \tag{6.4}$$

and the asymptotic variance of the estimator of ϕ is

$$\sigma_{11}\left[(\pi_{13}, \pi_{23}, \pi_{33})\mathbf{M}_{zz}^{0}\begin{pmatrix} \pi_{13} \\ \pi_{23} \\ \pi_{33} \end{pmatrix}\right]^{-1}. \tag{6.5}$$

If $n = 2$

$$\mathbf{B}_{\mathrm{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \phi & \psi \end{bmatrix}, \quad \mathbf{\Gamma}_{\mathrm{e}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{6.6}$$

and $\mathbf{B}_{\mathrm{E}}$ and $\mathbf{\Gamma}_{\mathrm{i}}$ are vacuous. Partition $\mathbf{P}$ as

$$\begin{bmatrix} p_{11} & p_{12} & p_{31} \\ p_{21} & p_{22} & p_{32} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = (\mathbf{p}_{\mathrm{e}1}, \mathbf{p}_{\mathrm{e}2}, \mathbf{p}_{\mathrm{e}3}) = \mathbf{P}. \tag{6.7}$$

The TSLS estimator of $(\phi, \psi) = \mathbf{A}$ is

$$-\frac{1}{\mathbf{p}'_{\mathrm{e}3}\mathbf{M}_{zz}\mathbf{p}_{\mathrm{e}3}}\,\mathbf{p}'_{\mathrm{e}3}\mathbf{M}_{zz}(\mathbf{p}_{\mathrm{e}1}, \mathbf{p}_{\mathrm{e}2}). \tag{6.8}$$

The asymptotic covariance matrix is

$$\mathbf{\Sigma}_{22} \otimes \left[(\pi_{13}, \pi_{23})\mathbf{M}^0_{zz}\begin{pmatrix} \pi_{13} \\ \pi_{23} \end{pmatrix}\right]^{-1}. \tag{6.9}$$

Note that the TSLS estimator of ϕ based on the first equation alone is one component of the TSLS estimator of (ϕ, ψ) based on two equations.

The estimator of $\mathbf{\Omega}$ is $\mathbf{W} = \frac{1}{T}(\mathbf{M}_{yy} - \mathbf{P}'\mathbf{M}_{zz}\mathbf{P})$. For $n = 1$ define $\lambda_1 > \lambda_2$ as the roots of

$$\left|\begin{pmatrix} \mathbf{p}'_{\mathrm{e}1} \\ \mathbf{p}'_{\mathrm{e}3} \end{pmatrix}\mathbf{M}_{zz}(\mathbf{p}_{\mathrm{e}1}, \mathbf{p}_{\mathrm{e}3}) - \lambda\begin{pmatrix} w_{11} & w_{13} \\ w_{31} & w_{33} \end{pmatrix}\right| = 0 \tag{6.10}$$

and $\mathbf{b}_2$ as the solution to

$$\left[\begin{pmatrix} \mathbf{p}'_{\mathrm{e}1} \\ \mathbf{p}'_{\mathrm{e}3} \end{pmatrix}\mathbf{M}_{zz}(\mathbf{p}_{\mathrm{e}1}, \mathbf{p}_{\mathrm{e}3}) - \lambda_2\begin{pmatrix} w_{11} & w_{13} \\ w_{31} & w_{33} \end{pmatrix}\right]\begin{pmatrix} b_{12} \\ b_{32} \end{pmatrix} = 0 \tag{6.11}$$

and $\mathbf{b}'_2\mathbf{W}'\mathbf{b}_2 = 1$. The LIML estimator of ϕ is $-b_{32}/b_{12}$.

To define the LIML estimator of $(\phi, \psi)'$ for $n = 2$ solve

$$|\mathbf{P}\mathbf{M}_{zz}\mathbf{P}' - \lambda\mathbf{W}| = 0 \tag{6.12}$$

for $\lambda_1 > \lambda_2 > \lambda_3$ and

$$(\mathbf{PMP}' - \lambda_{\mathrm{i}}\mathbf{W})\mathbf{b} = \mathbf{0}, \quad \mathbf{b}'\mathbf{W}\mathbf{b} = 1 \tag{6.13}$$

for $\mathbf{b}_{\mathrm{i}}$, $i = 1, 2, 3$. Let

$$\widetilde{\mathbf{B}} = (\mathbf{b}_2, \mathbf{b}_3) = \begin{pmatrix} \widetilde{\mathbf{B}}_1 \\ \widetilde{\mathbf{B}}_2 \end{pmatrix}. \tag{6.14}$$

Then

$$\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}_1^{-1} = \begin{bmatrix} \mathbf{I}_2 \\ \widetilde{\mathbf{B}}_2\widetilde{\mathbf{B}}_1^{-1} \end{bmatrix} \tag{6.15}$$

is the LIML estimator of $(\mathbf{I}_2, -\mathbf{A}')'$. The LIML estimator of ϕ for $n = 1$ is not a component of the LIML estimator for $n = 2$.

Example 2. Let

$$\begin{bmatrix} \mathbf{B} \\ \mathbf{\Gamma} \end{bmatrix} = \begin{bmatrix} 1 & \mu & 0 \\ \phi & 1 & 0 \\ 0 & 0 & \theta \\ \\ \alpha & 0 & \kappa \\ 0 & \rho & \delta \\ 0 & 0 & \tau \\ 0 & 0 & v \end{bmatrix}, \tag{6.16}$$

$\mathbf{y}_t = (y_{1t}, y_{2t}, y_{3t})'$, $\mathbf{z}_t = (z_{1t}, z_{2t}, z_{3t}, z_{4t})'$. The first equation is

$$\mathbf{y}_1 + \mathbf{y}_2\phi = \mathbf{z}_1\alpha + \mathbf{u}_1. \tag{6.17}$$

If $n = 1$, $\boldsymbol{\beta}_{\mathrm{I}}' = (1, \phi)$, $\boldsymbol{\beta}_{\mathrm{E}}' = 0$, $\gamma_{\mathrm{i}}' = \alpha$, $\gamma_{\mathrm{e}}' = (0, 0, 0)$; the relevant part of $\mathbf{P}$ (4×3) is

$$\begin{bmatrix} p_{21} & p_{22} \\ p_{31} & p_{32} \\ p_{41} & p_{42} \end{bmatrix} = (\mathbf{p}_{\mathrm{e}1}, \mathbf{p}_{\mathrm{e}2}). \tag{6.18}$$

Let $\mathbf{M}_{zz} = (m_{ij})$. Assume that $m_{12} = m_{13} = m_{14} = 0$ (for convenience). The TSLS estimator of ϕ is

$$-\frac{\begin{pmatrix} p_{22} & p_{32} & p_{42} \end{pmatrix} \begin{pmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} p_{21} \\ p_{31} \\ p_{41} \end{pmatrix}}{\begin{pmatrix} p_{22} & p_{32} & p_{42} \end{pmatrix} \begin{pmatrix} m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \\ m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} p_{22} \\ p_{32} \\ p_{42} \end{pmatrix}}. \tag{6.19}$$

If $n = 2$

$$\mathbf{B}_{\mathrm{I}} = \begin{bmatrix} 1 & \mu \\ \phi & 1 \end{bmatrix}, \quad \mathbf{B}_{\mathrm{E}} = (0, 0), \quad \mathbf{\Gamma}_{\mathrm{i}} = \begin{bmatrix} \alpha & 0 \\ 0 & \rho \end{bmatrix}, \quad \mathbf{\Gamma}_{\mathrm{e}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{6.20}$$

The block of equations is

$$(\mathbf{y}_1, \mathbf{y}_2)) \begin{bmatrix} 1 & \mu \\ \phi & 1 \end{bmatrix} = (\mathbf{z}_1, \mathbf{z}_2) \begin{bmatrix} \alpha & 0 \\ 0 & \rho \end{bmatrix} + (\mathbf{u}_1, \mathbf{u}_2). \tag{6.21}$$

A part of the reduced form is

$$(\mathbf{y}_1, \mathbf{y}_2) = (\mathbf{z}_1, \mathbf{z}_2) \begin{bmatrix} \alpha & -\alpha\mu \\ -\phi\rho & \rho \end{bmatrix} \frac{1}{1 - \mu\phi} + (\mathbf{v}_1, \mathbf{v}_2). \tag{6.22}$$

The least squares (LS) estimator of the coefficient matrix in (6.22) is

$$\frac{1}{1-\widehat{\mu}\widehat{\phi}}\begin{bmatrix} \widehat{\alpha} & -\widehat{\alpha}\widehat{\mu} \\ -\widehat{\phi}\widehat{\rho} & \widehat{\rho} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}. \tag{6.23}$$

Thus

$$\widehat{\alpha} = p_{11} - \frac{p_{12}p_{21}}{p_{22}}, \quad \widehat{\rho} = p_{22} - \frac{p_{12}p_{21}}{p_{11}}, \quad \widehat{\phi} = -\frac{p_{12}}{p_{22}}, \quad \widehat{\mu} = -\frac{p_{21}}{p_{11}}. \tag{6.24}$$

These estimators are LIML as well as TSLS.

7. Confidence regions and tests

When $\mathbf{V}$ in (1.1) or $\mathbf{U}$ in (4.1) is normally distributed, exact confidence regions for $\mathbf{B}$ or $\mathbf{B}_{\mathrm{I}}$ can be constructed. For $\mathbf{B}$ satisfying (1.3) $\mathbf{B}'\mathbf{P}'\mathbf{M}_{zz}\mathbf{PB}$ and $\mathbf{B}'\mathbf{WB}$ are independently distributed according to Wishart distributions $W_n(\mathbf{B}'\boldsymbol{\Omega}\mathbf{B}, k)$ and $W_n(\mathbf{B}'\boldsymbol{\Omega}\mathbf{B}, T-k)$, respectively. Then

$$\frac{|\mathbf{B}'\mathbf{WB}|}{|\mathbf{B}'\mathbf{P}'\mathbf{M}_{zz}\mathbf{PB} + \mathbf{B}'\mathbf{WB}|} = U_{n,k,T-k} \tag{7.1}$$

has the distribution of the criterion U, where n is the dimensionality, k is the number of columns of $\mathbf{PB}$ and $T-k$ is the number of degrees of freedom of $\mathbf{W}$. See Section 8.4 of Anderson (2003). This criterion is a monotonically increasing function of the likelihood ratio criterion for testing the null hypothesis that $\boldsymbol{\Pi}\mathbf{B} = \mathbf{0}$. Table B.1 of Anderson (2003) provides significance points.

A confidence region for $\mathbf{B}_{\mathrm{I}}$ consists of $G_{\mathrm{I}} \times n$ matrices $\mathbf{B}_{\mathrm{I}}^*$ such that

$$\frac{|\mathbf{B}_{\mathrm{I}}^{*\prime}\mathbf{W}_{\mathrm{II}}\mathbf{B}_{\mathrm{I}}^*|}{|\mathbf{B}_{\mathrm{I}}^{*\prime}\mathbf{P}_{\mathrm{eI}}'\mathbf{M}_{\mathrm{ee}}\mathbf{P}_{\mathrm{eI}}\mathbf{B}_{\mathrm{I}}^* + \mathbf{B}_{\mathrm{I}}^{*\prime}\mathbf{W}_{\mathrm{II}}\mathbf{B}_{\mathrm{I}}^*|} \geqslant U_{n,K_{\mathrm{e}},T-K}(\varepsilon). \tag{7.2}$$

Since the statistic in (7.2) is invariant with respect to multiplication of $\mathbf{B}_{\mathrm{I}}^*$ on the right by an $n \times n$ matrix, we normalize and identify $\mathbf{B}_{\mathrm{I}}^*$ by $\beta_{Lj}^* = 1$, $\beta_{\ell j}^* = 0$, $j = 1, \ldots, n$.

As an example, let $n = 2$ and $G_{\mathrm{I}} = 4$. Normalize and identify $\mathbf{B}_{\mathrm{I}}$ by $\beta_{11} = \beta_{22} = 1$ and $\beta_{12} = \beta_{21} = 0$. Then

$$\mathbf{B}_{\mathrm{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \beta_{31} & \beta_{32} \\ \beta_{41} & \beta_{42} \end{bmatrix}. \tag{7.3}$$

Then (7.2) provides a confidence region for β_{31}, β_{32}, β_{41}, and β_{42}.

The likelihood ratio criterion to test $H_0 : \mathbf{B} = \mathbf{B}^*$, where $\mathbf{B}^*$ is completely specified, is the left-hand side of (7.2). H_0 is rejected if the criterion is less than the significance point.

Other confidence regions and tests can be constructed by using other criteria for the multivariate linear hypothesis. See Anderson (2003, Section 8.6).

Appendix A. Example for Section 5.1

Let

$$\mathbf{E} = \begin{bmatrix} \mathbf{A} & \mathbf{c} \\ \mathbf{c}' & d \end{bmatrix} \tag{A.1}$$

be a positive definite matrix of order p. Let the roots of $|\mathbf{A} - \lambda \mathbf{I}_{p-1}| = 0$ be $\lambda_1 > \cdots > \lambda_{p-1}$ and the roots of $|\mathbf{E} - v\mathbf{I}_p| = 0$ be $v_1 > \cdots > v_p$. Let $\mathbf{b}$ satisfy

$$\mathbf{Ab} = \lambda_{p-1}\mathbf{b}, \tag{A.2}$$

and let $\mathbf{f}_j$ satisfy

$$\mathbf{Ef}_j = \begin{bmatrix} \mathbf{A} & \mathbf{c} \\ \mathbf{c}' & d \end{bmatrix} \begin{bmatrix} \mathbf{f}_{1j} \\ f_{pj} \end{bmatrix} = v_j \mathbf{f}_j, \quad j = 1, \ldots, p. \tag{A.3}$$

Is there a linear combination $\mathbf{h} = k_{p-1}\mathbf{f}_{p-1} + k_p\mathbf{f}_p$ such that for some $k \neq 0$

$$k\begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} = \mathbf{h} = k_{p-1}\begin{bmatrix} \mathbf{f}_{1,p-1} \\ f_{p,p-1} \end{bmatrix} + k_p \begin{bmatrix} \mathbf{f}_{1p} \\ f_{pp} \end{bmatrix} = \begin{bmatrix} k_{p-1}\mathbf{f}_{1,p-1} + k_p\mathbf{f}_{1p} \\ k_{p-1}f_{p,p-1} + k_p f_{pp} \end{bmatrix}? \tag{A.4}$$

If so, then the last component of (A.4) implies $k_{p-1}f_{p,p-1} + k_p f_{pp} = 0$ and $k\mathbf{b} \leqslant k_{p-1}\mathbf{f}_{1,p-1} + k_p\mathbf{f}_{1p}$. From (A.3) we derive

$$\begin{aligned} \mathbf{Eh} &= \begin{bmatrix} \mathbf{A} & \mathbf{c} \\ \mathbf{c}' & d \end{bmatrix} \begin{bmatrix} k_{p-1}\mathbf{f}_{1,p-1} + k_p\mathbf{f}_{1p} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A}(k_{p-1}\mathbf{f}_{1,p-1} + k_p\mathbf{f}_{1p}) \\ \mathbf{c}'(k_{p-1}\mathbf{f}_{1,p-1} + k_p\mathbf{f}_{1p}) \end{bmatrix} \\ &= k_{p-1}v_{p-1}\mathbf{f}_{p-1} + k_p v_p \mathbf{f}_p = \begin{bmatrix} k_{p-1}v_{p-1}\mathbf{f}_{1,p-1} + k_p v_p \mathbf{f}_{1p} \\ k_{p-1}v_{p-1}f_{p,p-1} + k_p v_p f_{pp} \end{bmatrix}. \end{aligned} \tag{A.5}$$

That is,

$$\mathbf{A}(k_{p-1}\mathbf{f}_{1,p-1} + k_p\mathbf{f}_{1p}) = k_{p-1}v_{p-1}\mathbf{f}_{1,p-1} + k_p v_p \mathbf{f}_{1p}. \tag{A.6}$$

However, if (A.4) holds, (A.6) is $\lambda_{p-1}(k_{p-1}\mathbf{f}_{1,p-1} + k_p\mathbf{f}_{1p})$, which is impossible since $\mathbf{f}_{1,p-1}$ and $\mathbf{f}_{1p}$ are linearly independent.

As an example for Section 5.1, let $p = G_{\mathrm{I}}$, $n = 2$, $\mathbf{A}$ be the effect matrix in (5.5), and $\mathbf{E}$ the effect matrix in (4.18). This shows that the identified block LIML estimator of an equation is not necessarily the same as the single-equation LIML estimator.

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